

Unconstrained Fitting of Non-Central Risk-Neutral Densities Using a Mixture of Normals

Riccardo Rebonato and Teresa Cardoso
QUARC

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Abstract

1 Statement of the Problem

One of the most important problems in calibrating option models (eg, stochastic-volatility, local-volatility, jump-diffusion, etc) is obtaining a reliable smile surface from the (often noisy and non-contemporaneous) market prices of plain-vanilla options. For some models it can be debated whether using 'undoctored' noisy prices might be better than regularizing the input, but, for some approaches, such as local-volatility models, having a smooth (and differentiable) input smile surface is a must (see the brief discussion below).

Many techniques have been proposed to obtain from the observed market prices of plain-vanilla options a smooth volatility surface. A systematic survey would require a full review article in itself. Without attempting to cover the literature systematically, we simply mention Shimko (1994) as a representative example of a fitting procedure directly to prices, Jacquier and Jarrow (1995), Madan, Carr and Chang (1998) and Elliot, Lahaie and Madan (1995) as examples of fitting to transformed prices, Ait-Sahalia (2002) for a direct fitting of the implied volatility curve, Avellaneda (1998) for an application of the minimum-entropy methodology, Mirfendereski and Rebonato (2001) for direct modelling of the risk-neutral density. Most of these models fall in either of three categories: smooth interpolation of (possibly transformed) prices, smooth interpolation of implied volatilities, or smooth fitting of risk-neutral densities.

We argue that modelling directly the density is the most desirable approach, because a smooth and 'plausible' density will certainly produce (by integration) smooth and well-behaved price and implied volatility surfaces, but the converse is not true. See, for instance, the discussion in Mirfendereski and Rebonato

(2001). Indeed, if one models directly prices one can easily obtain wildly oscillating densities because the latter are obtained, modulo discounting, from the call prices by double differentiation:

$$\phi(S) = \frac{\partial^2 C}{\partial K^2} \quad (1)$$

It is important to observe that no model-dependent assumptions are needed to obtain Equation 1: one simply requires that the price of a European option can be written as a discounted expectation of the terminal payoff, G , over *some* risk-neutral density.

Willmott (1998) highlights that, if the implied risk-neutral density is not well behaved, local-volatility solutions can become extremely noisy. This is easy to see, if one recalls Dupire's (1994) well-known result

$$\sigma_{K,T}^2 = 2 \frac{\frac{\partial C_{K,T}(0,S)}{\partial T} + (r-d)K \frac{\partial C_{K,T}(0,S)}{\partial K} + dC_{K,T}(0,S)}{\frac{\partial^2 C_{K,T}(0,S)}{\partial K^2} K^2} \quad (2)$$

Note, in fact, that the denominator in the expression above is simply linked to the risk-neutral density. Rebonato (2003) argues that obtaining smooth and well-behaved density functions is important also in more general modelling contexts.

One might argue that, as long as the implied volatility has been smoothly interpolated, the resulting risk-neutral densities will be smooth as well. Figs 1 and 2 show that this is not the case: Fig. 1 displays two optically indistinguishable implied volatility curves, and Fig 2 the associated risk-neutral densities. It is clear that relying on the smoothness of the implied volatility curve would not guarantee, for instance, that the local volatility extracted using Equation 2 would have desirable properties.

This paper therefore sets as its goal to obtain an excellent fit to the market prices of plain-vanilla options by directly modelling the risk-neutral density. We choose to do so by extending the mixture-of-normals approach presented in Alexander (2001). We present a numerical technique that will allow us to find the solutions in an efficient and unconstrained manner, while automatically satisfying all the financial and mathematical requirements.

2 Summary of Known Results

If one wants to find the risk-neutral density, $\phi(\ln S_T)$, implied by a set of market prices for a given time horizon T , one can express it as a linear combination of normal densities, $\varphi(\mu_i, \sigma_i^2)$:

$$\phi(\ln S_T) = \sum_i w_i \varphi(\mu_i, \sigma_i^2) \quad (3)$$

with the normalization condition

$$\sum_i w_i = 1 \quad (4)$$

(This is the approach followed, for instance, by Alexander (2001)). The mean and the variance of the mixture, μ_ϕ and σ_ϕ^2 , are related to the means and variances of the original normal distribution by the relationships

$$\mu_\phi = \sum_i w_i \mu_i \quad (5)$$

$$\sigma_\phi^2 = \sum_i w_i \sigma_i^2 + \left[\sum_i w_i \mu_i^2 - \left(\sum_i w_i \mu_i \right)^2 \right] \quad (6)$$

Note carefully that, since one is fitting risk-neutral densities, the first moment is not a free-fitting parameter, but must recover the forward condition, ie the expectation in the risk-neutral measure of the 'stock' price must equal its forward value. Very often this condition has been enforced in the literature (see, eg, Alexander (2001)), by requiring that all the μ_i should be equal to the risk-neutral drift. This is, however, unnecessarily restrictive, because it forces the distribution of log-prices to display no skew. We show below how this feature can be naturally incorporated in the method we propose.

Since one of the most common features of empirical distributions is their leptokurtosis, it is also useful to give an expression (see Alexander (2001)) for the excess kurtosis for a mixture of normals. In the case when all the terms μ_i are be equal to the risk-neutral drift one obtains:

$$\kappa_\phi = 3 \left[\frac{\sum_i w_i \sigma_i^4}{\left(\sum_i w_i \sigma_i^2 \right)^2} - 1 \right] \quad (7)$$

From Equation 7 it is clear that the density of a mixture of normals (with same means μ_i !) will always have a positive excess kurtosis (ie, will be more leptokurtic than a normal density). This is because, for any non-degenerate case,

$$\sum_i w_i \sigma_i^4 > \left(\sum_i w_i \sigma_i^2 \right)^2 \quad (8)$$

If one wanted to use a mixture of normals to fit an empirical price densities there are therefore two main routes:

1. one can estimate the first four moments of the empirical distribution of the logarithms of the price density, select two normals as the 'basis set' and fit the four moments exactly. With the fitted mixture-of-normals distribution the prices for the calls, C_K^T , can be determined and compared with the market values. The procedure is very straightforward, but the fit to the market prices is unlikely to be very good.
2. one can determine the optimal weights $\{w\}$ by means of a least-square fit to the option prices after converting the density into call prices. This procedure is made easy by the fact that the option price C_K^T is simply given by a linear combination with the same weights $\{w\}$ of Black-and-Scholes formulae.

As mentioned above, however, if all the normal densities in the mixture are 'centered' (in log space) around the forward value, one is automatically guaranteed to recover the no-arbitrage forward pricing condition, but the resulting pricing density will display no skewness. This is at odds with empirical findings (see, eg, Madan et al). Skewness can be easily obtained by allowing the different constituent Gaussian densities to be centered around different location coefficients, μ_i . By so doing, however, some care must be given to recovering the first moment of the density exactly, since this is linked to the no-arbitrage cash-and-carry forward condition. Furthermore, if the weights are left unconstrained (apart from Equation 4), there is no guarantee that the resulting density will be everywhere positive. The following section shows how both these problems can be overcome.

3 Fitting the Risk-Neutral Density Function: Mixture of Normals

3.1 Ensuring the Normalization and Forward Constraints

Denote by S_i the price of the stock at time T_i : $S(T_i) = S_i$. If we denote by $\Phi(S_i)$ its risk-neutral probability density, we want to write

$$\Phi(S_i) = \sum_k w_k^i \varphi(S_k^i) \quad (9)$$

where

$$\varphi(S_k^i) = \mathcal{LN}(\mu_{ik}, \sigma_{ik}^2; S_0) \quad (10)$$

and $\mathcal{LN}(\mu_{ik}, \sigma_{ik}^2; S_0)$ denotes a log-normal density with

$$E(S_k^i) = S_0 \exp(\mu_{ik} T_i) \quad (11)$$

$$Var(S_k^i) = [S_0 \exp(\mu_{ik} T_i)]^2 [\exp(\sigma_{ik}^2 T_i) - 1] \quad (12)$$

By this expression, the risk-neutral¹ density for the stock price is expressed as a sum of log-normal densities, and therefore the stock price density is *not* log-normal.

In order to ensure that the density is everywhere positive, we require that all the weights should be positive. This can be achieved by imposing:

$$w_k^i = (\alpha_k^i)^2 \quad (13)$$

The normalization condition, which requires that

$$\sum_k w_k^i = 1 \quad (14)$$

¹To lighten the prose, the qualifier 'risk-neutral' is often omitted in the following where there is no risk of ambiguity.

therefore becomes

$$\sum_k (\alpha_k^i)^2 = 1 \quad (15)$$

This condition can always be satisfied by requiring that the coefficients α_k^i should be the polar co-ordinates of a unit-radius hyper-sphere. Therefore we can write

$$\alpha_k^i = f(\theta_1^i, \theta_2^i, \dots, \theta_{n-1}^i) \quad (16)$$

For instance, for $n = 2$ one simply has

$$\alpha_1^i = \sin(\theta_1^i) \quad (17)$$

$$\alpha_2^i = \cos(\theta_1^i) \quad (18)$$

This is certainly acceptable, because, for *any* angle θ_1^i ,

$$\sin(\theta_1^i)^2 + \cos(\theta_1^i)^2 = (\alpha_1^i)^2 + (\alpha_2^i)^2 = 1 \quad (19)$$

and equation 15 is satisfied. In the more general case the coefficients α_k^i are given by:

$$\alpha_k^i = \cos \theta_k^i \prod_{j=1}^{k-1} \sin \theta_j^i \quad k = 1, 2, \dots, m-1 \quad (20)$$

$$\alpha_k^i = \prod_{j=1}^{k-1} \sin \theta_j^i \quad k = m \quad (21)$$

The reason for expressing the coefficients α_k^i in terms of 'angles' is that we will want to optimize the model density over the weights w_k^i in an *unconstrained* manner, while automatically resting assured that the resulting linear combination is a possible density. This will be the case only if 15 is always satisfied. In general, ie, if one tried to optimize directly over the weights w_k^i , one would have to carry out a heavily constrained numerical search: not only every weight w_k^i would have to be greater than zero but smaller than one, but also every partial sum over the weights would have to be strictly positive² and less than one. The procedure suggested above automatically ensures that this will always be the case, and therefore allows one to carry out an *unconstrained* optimization over the angle(s) θ .

Apart from the requirements in Equation 15, there is at least one more constraints. The no-arbitrage forward condition

$$E[S_i] = \int \Phi(S_i = \tilde{S}_i | S_0 = \hat{S}) \tilde{S}_i d\tilde{S}_i = S_0 \exp(rT_i) \quad (22)$$

must in fact always be satisfied exactly under penalty of arbitrage³. This could be trivially achieved by imposing

$$\mu_{ik} = r \quad \text{for any } k \quad (23)$$

²If, in the real-world measure, we assume that all positive values of the underlying are possible, the density cannot go to zero under any *equivalent* measure.

³We denote by r either the short rate or the difference between the short rate and the dividend yield or the difference between the domestic and the foreign short rates, according to whether one is dealing with the case of a non-dividend paying asset, of a dividend stock or of an FX rate, respectively.

This, however, would give rise to densities with kurtosis but no skew. (Indeed, this is the approach suggested by Alexander (2001)). To obtain skew, we want to allow the various basis functions to be centred around different locations in log S -space, but we want to do so while retaining the forward-pricing condition. This can be achieved as follows. From the relationships above we can write

$$E[S_i] = E\left[\sum_k w_k^i S_i^k\right] = \sum_k w_k^i E[S_i^k] = S_0 \exp(rT_i) \quad (24)$$

Recalling that

$$E[S_k^i] = S_0 \exp(\mu_{ik}T_i) \quad (25)$$

one finds that

$$\exp(rT_i) = \sum_k w_k^i \exp(\mu_{ik}T_i) \quad (26)$$

The summation over the number of basis functions, k , can be split into the first term, and the sum, \sum' , over the remaining terms:

$$\exp(rT_i) = w_1^i \exp(\mu_{i1}T_i) + \sum_k' w_k^i \exp(\mu_{ik}T_i)$$

which can be solved for μ_{i1} :

$$\begin{aligned} \exp(\mu_{i1}T_i) &= \frac{\exp(rT_i) - \sum_k' w_k^i \exp(\mu_{ik}T_i)}{w_1^i} \rightarrow \\ \mu_{i1} &= \frac{\ln\left[\frac{\exp(rT_i) - \sum_k' w_k^i \exp(\mu_{ik}T_i)}{w_1^i}\right]}{T_i} \end{aligned} \quad (27)$$

In other words, if, for any maturity T_i , we choose the first location coefficient according to the expression above, we can always rest assured that the forward condition will be automatically satisfied. Note, however, that, *a priori*, there is no guarantee that the argument of the logarithm will always be positive. In practice, we have never found this to be a problem.

So, for any set of angles $\{\theta\}$, for any set of σ_{ik} , $k = 1, 2, \dots, n$ and for any set of μ_{ik} , $k = 2, 3, \dots, n$ the forward and the normalization conditions will always be satisfied if μ_{i1}^i is chosen to be given by Equation 27. In the following we will always assume that this choice has been made.

3.2 The Fitting Procedure

Let $Call_{K_j}^{T_i}(mod)$ be the model value of the call expiring at time T_i for strike K_j , and $Call_{K_j}^{T_i}(mkt)$ the corresponding market prices. The quantity $Call_{K_j}^{T_i}(mod)$ is given by

$$Call_{K_j}^{T_i}(mod) = \sum_{k=1,n} \alpha_k^i(\theta)^2 \int \varphi(S_i^k) G(S_i^k, K_j) dS_i^k \quad (28)$$

where $G(S_i, K_j)$ is the pay-off function:

$$G(S_i^k, K_j) = [S_i^k - K_j]^+ \quad (29)$$

and $\varphi(S_i^k)$ is the log-normal density:

$$\varphi(S_i^k) = \frac{1}{S_i^k \sigma_{ik} \sqrt{2\pi T_i}} \exp \left[-\frac{1}{2} \left(\frac{\ln\left(\frac{S_i^k}{S_0 \exp(\mu_{ik} T_i)}\right) + \frac{1}{2} \sigma_{ik}^2 T_i}{\sigma_{ik} \sqrt{T_i}} \right)^2 \right] \quad (30)$$

Note that each term under the integral sign is simply equal to the value of a Black-and-Scholes call when the 'riskless rate' is equal to μ_{ik} and the volatility is equal to σ_{ik} . Therefore one can write:

$$Call_{K_j}^{T_i}(mod) = \sum_{k=1, n} \alpha_k^i(\theta)^2 Call_{BS}(\mu_{ik}, K_j, T_i, \sigma_{ik}) \quad (31)$$

where μ_{i1} is fixed from the previous forward relationship. Equation 31 lends itself to a simple interpretation: for any given strike, K_j , the model price is expressed as a linear combination of Black-and-Scholes prices, with the same strike, time to maturity and volatility, but with riskless rate equal to μ_{ik} and volatility equal to σ_{ik} .

Now define χ^2 as

$$\chi^2 = \sum \left[Call_{K_j}^{T_i}(mod) - Call_{K_j}^{T_i}(mkt) \right]^2 \quad (32)$$

Then, in order to obtain the optimal fit to the observed set of market prices we simply have to carry out an *unconstrained* minimization of χ^2 over the $(n-1)$ angles $\{\theta\}$, the n volatilities $\sigma_{ik}, k = 1, 2, \dots, n$, the $(n-1)$ location coefficients $\mu_{ik}, k = 2, 3, \dots, n$ and with μ_{1i} given by (27). Therefore, for each expiry we have at our disposal $3n-2$ coefficients. (For $n=1$, we simply have one coefficient, ie one volatility. For $n=2$ we have four coefficients, ie, two volatilities, one weight (ie one angle θ), and one location coefficient; etc).

4 Numerical Results

4.1 Description of the Numerical Tests

In this section we explore how well the mixture-of-normals method works in practice. We do so by looking both at theoretical densities and at market prices. The theoretical densities are obtained from three important models that will be discussed in the following, ie the jump-diffusion, the stochastic-volatility and the variance-gamma model. We have sometimes used rather 'extreme' choices of parameters in order to test the robustness and flexibility of the approach.

Finally, for simplicity we assumed a non-dividend-paying stock (interest rate at 5%) with spot price of \$100, and we looked at maturities of 0.5, 1, 2 and 4

years. Longer maturities, because of the Central Limit Theorem, would actually produce an easier test. All the optimized coefficients are reported in Tabs I to III. As for the market prices, they were obtained from the GBP\$ caplet market in March 2003.

Tab I: The means, standard deviations and weights obtained for the fits to the jump-diffusion, stochastic-volatility and variance gamma models discussed in the text using a mixture of three log-normals. The chi squared statistics is also displayed.

Tab II: Same as Tab. I for the mixture of four log-normals in the jump-diffusion case

Tab III: Same as Tab II for the market data used (GBP\$ caplet implied volatilities, March 2003)

4.2 Fitting to Theoretical Prices: Stochastic-Volatility Density

The simplest test is probably that of a stochastic-volatility process for the underlying, because we know that, in this case, the process for the logarithm of the price directly generates a terminal risk-neutral density which is made up of a mixture of normals⁴. A mean-reverting process was chosen for the volatility, with an initial value of the volatility equal to the reversion level (12.13%). The volatility of the volatility was given a very high value (100%) to 'stress' the test. We assumed no correlation between the Brownian shocks affecting the underlying and the volatility.

The smiles produced by these parameters are shown in Fig 3. The theoretical density, the fitted density and the log-normal density matched to the first two moments are shown in Fig. 5. The match is excellent everywhere even with just three basis functions. The resulting fit to the smile is shown in Fig. 4.

Fig 3: The smiles produced by the parameters discussed in the text in the stochastic-volatility case.

Fig 4: The fit to the 1-year stochastic-volatility smile obtained using three log-normals.

Fig 5: The theoretical density, the fitted density and the moment-matched log-normal density in the stochastic-volatility case

It is interesting to point out that we found that, after optimization, the three basis distributions in the mixture turned out to have the same (risk-neutral) mean even if they were not required to be so centered. This is consistent with

⁴Each normal density component would have as variance the square of the root-mean-squared volatility encountered along each volatility path.

the assumption of the underlying following a stochastic-volatility process, which automatically produces a mixture of identically-centered distributions. This results in a non-skewed density (see Figure 5). On the other hand, the resulting distribution has positive kurtosis, correctly reproduced by the fitting procedure.

4.3 Fitting to Theoretical Prices: Variance-Gamma Density

For this model, we consider the whole set of parameters estimated by Madan, Carr and Chang (1998) for the risk-neutral density of the S&P: $\sigma=12.13\%$, $\nu = 16.86\%$, $\theta = -0.1436$. Madan, Carr and Chang (1998) show that, in the risk-neutral world, the hypothesis of zero skewness can be rejected. Their risk-neutral density will therefore provide the first test for our method when the underlying distribution is skewed. Fig 8 displays the fit to the 2-year density obtained with just three basis functions, together with a moment-matched log-normal fit. Fig 7 displays the fit to the two-year smile. It is clear from the figure that the model prices are everywhere recovered well within bid-offer spread⁵.

Fig 6: The smiles produced by the parameters discussed in the text in the variance-gamma case

Fig 7: The fit to the 2-year variance-gamma smile obtained using three log-normals.

Fig 8: The theoretical density, the fitted density and the moment-matched log-normal density in the variance-gamma case

4.4 Fitting to Theoretical Prices: Jump-Diffusion Density

The last theoretical smile we consider is the 'stress case' of a log-normal jump-diffusion process with parameters chosen so as to produce a multi-modal risk-neutral density for some maturities. Under what circumstances a jump-diffusion process can give rise to multi-modal densities, and what this implies for the associated smiles is discussed in Rebonato (2003). The model parameters used were 1 jump/year for the jump frequency, and an expectation and volatility of the jump amplitude ratios of 0.7 and 1%, respectively. The volatility of the diffusive part was taken to be constant at 12.13%.

The theoretical density (expiry 0.5 years) is shown in Fig 11 with a thin continuous line. The same figure also shows for comparison a moment-matched log-normal density. The line with markers then shows the risk-neutral density obtained with a mixture of three log-normals. It is clear that, even for such a difficult-to-match theoretical risk neutral density, a very good agreement is obtained everywhere (with the exception of the very-low-strike region) with as few as three log-normals. Fig 12 shows that the fit to the risk-neutral density

⁵The bid-offer spread was assumed to be half a vega (50 basis points in volatility).

becomes virtually perfect everywhere with 5 basis functions. The resulting theoretical and fitted smiles for expiry 0.5 years (the most challenging one) are shown in Fig 10. One can observe that everywhere the target and fitted implied volatilities coincide to well within bid-offer spread. The largest discrepancy was found to be 0.7 basis points in volatility (in these units one vega would be 100 basis points).

Fig 9: The smiles produced by the parameters discussed in the text in the jump-diffusion case

Fig 10: The fit to the 0.5-year jump-diffusion smile obtained using three log-normals.

Fig 11: The theoretical density, the fitted density and the moment-matched log-normal density in the jump-diffusion case (three lognormals). Note the relatively poor recovery of the theoretical density in the far left tail.

Fig 12: The theoretical density, the fitted density and the moment-matched log-normal density in the jump-diffusion case (four lognormals). The density is now well recovered everywhere.

4.5 Fitting to Market Prices

In order to test the method with real market prices we looked at the smile curves for caplet prices (GBP, March 2003) for different expiries. See Fig. 13. Again, the shortest maturity provided the most challenging test. We show the fits obtained for all the maturities (0.5, 1, 2, 4 and 8 years). Also in this case, the fit is virtually perfect everywhere with four log-normal basis functions.

On the basis of these results, one can conclude that the mixture-of-normals approach provides a simple and robust method to fit even very complex price patterns. The fact that the model price is expressed as a linear combination of Black-and-Scholes prices makes it very easy to calculate the derivative $\frac{\partial C}{\partial S}$. It also makes it very tempting to interpret this derivative as the 'delta' statistic, ie, as the amount of stock that will allow to hedge (instantaneously) against movements in the underlying, and to replicate a payoff by expiry. This interpretation is however unwarranted, as we discuss below.

Fig 13: The market caplet smiles for several maturities (GBP, March 2003)

Fig 14 to 18: The fit to the market data using four log-normals for expiries of 0.5, 1, 2, 4, and 8 years

5 Is the Term $\frac{\partial C}{\partial S}$ Really a Delta?

When one uses a fitting procedure like the mixture-of-normals approach one expresses the marginal (unconditional) price densities in terms of one or more basis functions. The terminal densities therefore display a parametric dependence on the value of the 'stock' price today, S_0 . Since the plain-vanilla option prices (denoted by C for brevity in this section) are obtainable as integrals over these probability densities, also these prices will display a parametric dependence on the initial value of the stock. It is therefore possible to evaluate the quantity $\frac{\partial C}{\partial S}$. Furthermore, in the mixture-of-normal case one can do so analytically. If one wanted to, one could also evaluate $\frac{\partial C^2}{\partial S^2}$, $\frac{\partial C}{\partial t}$, etc. This has led to the statement, often found in the literature, that these closed-form expressions for the price, the 'delta', the 'gamma' etc of the plain-vanilla option constitute an alternative pricing model, different from, but on a conceptual par with, say, the Black-and-Scholes model. Some authors speak of 'pricing' and 'hedging' with the mixture-of-normals approach, and refer to the pricing of options as the 'mixture models'. This however should be qualified very carefully. This is because the quantity $\frac{\partial C}{\partial S}$ plays in the Black-and-Scholes world not just the role of the derivative of the call price with respect to the stock price today, but also of the amount of stock we have to hold in order to be first-order neutral to the stochastic stock price movements. So, in the Black-and-Scholes world we *posit* a process for the underlying, and we *obtain* that, if this process is correct, we can create a risk-less portfolio by holding a $\frac{\partial C}{\partial S}$ amount of stock. By following this delta-neutral strategy to expiry we can replicate the terminal payoff. It is the replication property that allows us to identify the fair price of the option with the price of the replicating portfolio. The delta quantity is however only correct only because we have assumed that the process was known (a geometric diffusion in the Black-and-Scholes case). This is in general not true for the same quantity $\frac{\partial C}{\partial S}$ when seen in the context of the fitting procedures presented above. This is because the market prices, which our recover could recover almost perfectly, might have been produced by a more complex process (perhaps by a process that does not allow perfect replication by dealing in the underlying). If we want, we can still call the terms $\frac{\partial C}{\partial S}$, $\frac{\partial C^2}{\partial S^2}$, $\frac{\partial C}{\partial t}$ 'delta', 'gamma', 'theta' etc, but their financial interpretation in terms of a self-financing dynamic trading strategy is not warranted. The problem (see, eg, Piterbarg (2003)) is that the process for the underlying is not uniquely specified by the marginal densities. So, even if these were perfectly recovered by the fitting procedure, the trader would still not know what process has generated them. (Note that Brigo and Mercurio (2000) circumvent this problem by carrying out the mixture-of-normals fitting in conjunction with a local-volatility approach, that is designed to recover exactly any exogenous set of option prices).

6 Conclusions

We have shown the effectiveness of mixture of normals as basis functions in terms of which a great variety of risk-neutral densities can be expanded. The contribution of this paper is to show how the positivity and forward-pricing conditions can be automatically and simply satisfied without requiring that all the basis Gaussian functions should have the same mean. This would imply (in log space) absence of skew, which, at least in the risk-neutral measure, is a well-established and essential feature of the market smile.

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