Rapid computation of prices and deltas of nth to default swaps in the Li Model

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Summary

• Basic description of an nth to default swap
• Introduction to the Li model
• Solutions: Importance Sampling
• Parameter hedging and why computing sensitivities are difficult.
• Solutions: Likelihood & Pathwise Methods
• Results
Nth to default swaps: Product definition

• In an nth default swap a regular fee is paid until \( n \) of a basket of \( N \) credits have defaulted, or the deal finishes.

• When the Nth default occurs a payment of \( 1 - R \) is made to the fee payer.

\( R = \) recovery rate of \( n \)th defaulting asset
Nth to default swaps: Product definition

a. $n$ th Default occurs

\[ \text{Principal plus accrued interest} \]

\[ D_n(\tau_1, \ldots, \tau_N) \]

\[ \text{Spreads} \]

\[ \text{Recovery Rate} \]

b. $n$ th Default does not occur

\[ T_1^{SP} \]

\[ T_2^{SP} \]

\[ T_3^{SP} \]

\[ \ldots \]

\[ T \]

\[ S_1 \]

\[ S_2 \]

\[ S_3 \]

\[ \ldots \]

\[ \text{Spreads} \]
The Li Model

- Defaults are assumed to occur for individual assets according to a Poisson process with a deterministic intensity called the *hazard rate*.

- This means that default times are exponentially distributed.

- Li: Correlate these default times using a Gaussian copula
Consider some security A. We define the default time, $\tau_A$, as the time from today until A defaults.

We assume the defaults to occur as a Poisson process.

The intensity of this process, $h(t)$, is called the hazard rate.
Given a correlation matrix $C$ we compute $A$ such that

$$AA^T = C$$

Let $E(\tau, h)$ denote the cumulative exponential distribution function in $\tau$ given a fixed $h$:

$$E(\tau, h) = \mathbb{P}(t < \tau) = 1 - \exp(-\int_0^\tau h_j(t)dt).$$

$E^{-1}(u, h)$ denotes its inverse for fixed $h$. 

The Pricing Algorithm: SetUp
The Pricing Algorithm

- Draw a vector of independent normals, $z$
- Generate a set of correlated Gaussian deviates:
  $$w = Az.$$  
- Map to uniforms:
  $$u_i = N(w_i)$$
- Map to default times:
  $$\tau_i = E^{-1}(u_i, h)$$
- Compute the cash flow in this scenario; discount back.
  $$F(\tau_1, \ldots, \tau_N) = P(D_n(\tau_1, \ldots, \tau_N))[V_{prot} + (1 - r_n)H(T - D_n(\tau_1, \ldots, \tau_N))].$$
Importance Sampling

Intuitively: want to sample more thoroughly in the regions where defaults occur.

Look at a $k$th to default swap:

- Product pays a constant amount unless $k$ defaults occur.
- Restrict our attention to cases of $k$ defaults.
- By subtracting the constant, we can assume value is zero unless $k$ defaults occur.
Importance Sampling

• General Strategy: alter the probabilities of default such that we always get $k$ defaults. Each path is then "important"; compute prices.

• We then reweight the different contributions according to our changes to the probability measure.
Designing the importance density when $i = 1$

- Make the $i$th asset default before $T$ with probability:

\[
\frac{1}{(n + 1) - i}
\]

- Why? After $i$ non defaults want all the remaining credits to have an equal chance of default

- Pick a uniform $u_i$. If:

\[
u_i < \frac{1}{n + 1 - i}\]

map $u_i$ to a region where asset $i$ defaults.

\[
u_i > \frac{1}{n + 1 - i}\]

map $u_i$ a region where asset $i$ doesn’t default.
Designing the importance density when $i = 1$
Designing the importance density

\[ \frac{1}{n + 1 - i} \]

• Look at the original default region for asset \( i \)

\[ \tau_i < T \quad \rightarrow \quad w_i < x \quad \rightarrow \quad \sum_j A_{ij} z_j < x \]

Correlated Gaussian

• For our first to default case:

\[ a_{11} z_1 < x \quad \Rightarrow \quad z_1 < \frac{x}{a_{11}} \]

• Translate to uniforms:

\[ p_1 = N\left(\frac{x}{a_{11}}\right) \]
First to Default occurs:

- $u_1$ maps to $v_1$ where: 
  \[
  \frac{v_1}{p_1} = u_1 n \implies v_1 = u_1 n p_1
  \]

First to Default doesn’t occur:

- $u_1$ maps to $v_1$ where: 
  \[
  v_1 = p_1 + \frac{1 - p_1}{1 - \frac{1}{n}} (u_1 - \frac{1}{n})
  \]
We need to scale the contributions of these paths

First asset defaults: weight by \( np_1 \)

Doesn’t default: weight by \( \frac{1 - p_1}{1 - \frac{1}{n}} \)

Suppose that we have dealt with the first \((j - 1)\) assets. The unmassaged default probability now depends on \( Z \):

\[
W_j < x_j \text{ if and only if } \sum_{i < j} a_{ij} Z_i + a_{jj} Z_j < x_j.
\]

However, as \( A \) is lower triangular we have

\[
p_j = \frac{x_j - \sum_{i < j} a_{ij} Z_j}{a_{jj}}
\]

And repeat as before.
Computing Hazard Rate Sensitivities

• We hedge against changes in the hazard rates of the individual assets using “vanilla” default swaps.

• Naïve methods for determining hazard rate sensitivities (finite differencing)

\[ \Delta = \frac{P(h_i + \epsilon) - P(h_i)}{\epsilon} \quad \text{or} \quad \Delta = \frac{P(h_i + \epsilon) - P(h_i - \epsilon)}{2\epsilon} \]

have severe limitations due to their (very) slow rate of convergence.
First to default, 4 credits, 2 year deal Not a stress case!

Computing Hazard Rate Sensitivities
Computing Hazard Rate Sensitivities

Fourth to default, 4 credits, 0.15 year deal

![Graph showing hazard rate sensitivities with two methods: Centred Differencing and Pathwise Importance Sampling. The x-axis represents the number of paths, ranging from 256 to 4,194,304, and the y-axis shows hazard rate sensitivity. The graph compares the two methods visually.](image-url)
Why is Bumping problematic?

• Very few paths will give multiple defaults a short time (e.g., 0.15 years). If obligors are uncorrelated,

\[
\text{Prob } n \text{ defaults } = (hT)^n
\]

We therefore need lots of paths, even for pricing.

• When we compute sensitivities, bump one hazard rate. Very small change in the number of paths which now have \( n \) defaults compared to previously.
A CDS is similar to a barrier option, pay-out jumps according to whether Nth default is before or after deal maturity.

Value CDS =

$$\int P(D_n(\tau_1, \ldots, \tau_N))[(1-r_n)H(T-D_n(\tau_1, \ldots, \tau_N))\psi(\tau_1, \ldots, \tau_N)]d\tau_1 \ldots d\tau_N.$$ 

When we differentiate the payoff w.r.t the hazard rates we get a $\delta$ function.

Sampling this by Monte Carlo is very hard.
Well-known techniques for computing Greeks by Monte Carlo include:

- **Likelihood ratio**: differentiate the probability density function analytically, inside the integral.

- **The Pathwise Method**: differentiate the Payoff. Generally believed not to apply to discontinuous payoffs – we show that it does apply. Broadie-Glasserman

- **Malliavin calculus**: differentiation w.r.t. the underlying Brownian motion; not applicable here.
Value of the option:

\[ V = \mathbb{E}^Q[F(S_T)] = \int F(S)\psi(S, \theta) \, dS \]

We can write the sensitivity w.r.t \( \theta \):

\[ \frac{\partial V}{\partial \theta} = \int F(S) \frac{\partial}{\partial \theta} \psi(S, \theta) \, dS \]

No longer integrating against our Monte Carlo density! However, we can reintroduce it:
To compute sensitivity we reweight the payoff with:

$$\frac{\partial V}{\partial \theta} = \int F(S) \frac{\partial \psi(S, \theta)}{\partial \theta} \frac{1}{\psi(S, \theta)} \psi(S, \theta) \, dS$$

$$= \int F(S) \frac{\partial}{\partial \theta} \log \psi(S, \theta) \psi(S, \theta) \, dS$$

:. To compute sensitivity we reweight the payoff with:

$$\frac{\partial}{\partial \theta} \log \psi(S, \theta)$$
The Pathwise Method

• The delta of an option with payoff $F(S_T)$ is:

$$\Delta = \frac{\partial V}{\partial S_0} = e^{-rT} \int F(S_T) \frac{\partial}{\partial S_0} \psi(S_T, S_0, \ldots) \, dS_T$$

• For the case of a lognormal evolution we can show:

$$\Delta = \frac{\partial \psi}{\partial S_0} = -\frac{\partial}{\partial S_T} \left( \frac{S_T}{S_0} \psi \right)$$

• Integrating by parts and eliminating the boundary term:

$$\Delta = e^{-rT} \int \frac{\partial F(S_T)}{\partial S_T} \frac{S_T}{S_0} \psi(S_T, S_0, \ldots) \, dS_T$$
The Pathwise Method

• We are now differentiating the *payoff*!

• Suppose we have a digital option:
  \[ f(S_T) = H(S_T - K) \]

• Differentiate and we get a \( \delta \) function
The Likelihood Ratio Method for nth Default Swaps

• Value of the CDS:

\[
\int P(D_n)(1-r_n)H(T-D_n)\psi(\tau_1, \ldots, \tau_N)d\tau_1 \ldots d\tau_N.
\]

• Differentiate w.r.t. \(i\)th hazard rate:

\[
\frac{\partial V}{\partial h_i} = \int_0^T P(D_n)(1-r_n)H(T-D_n) \frac{\partial \psi(\tau_1, \ldots, \tau_N)}{\partial h_i} d\tau_1 \ldots d\tau_N.
\]

• Applying Broadie/Glasserman’s trick:

\[
\frac{\partial V}{\partial h_i} = \int_0^T P(D_n)(1-r_n)H(T-D_n) \frac{\partial \log \psi(\tau_1, \ldots, \tau_N)}{\partial h_i} \psi(\tau_1, \ldots, \tau_N)d\tau_1 \ldots d\tau_N.
\]
The calculation is straightforward for Gaussian copula and flat hazard rates:

\[
\frac{\partial \log \psi(\tau_1, \ldots, \tau_n)}{\partial h_i} = -(\rho^{-1} - 1)_{ij} \eta_j \frac{\partial \eta_i}{\partial u_i} \frac{\partial u_i}{\partial h_i} + \frac{1}{h_i} - \tau_i
\]

where \( \rho \) is the correlation matrix and

\[
\eta_i = \phi^{-1}(u_i) \quad \frac{\partial \eta_i}{\partial u_i} = \sqrt{2\pi e} \phi^{-1}(u_i)^2
\]
The Pathwise Method for nth Default Swaps

- We differentiate the discounted pay-off w.r.t $h_j$ (ignore the spreads for the moment):

$$F(\tau_1, \ldots, \tau_N) = P(D_N(\tau_1, \ldots, \tau_N))[(1-r_n)H(T-D_n(\tau_1, \ldots, \tau_N))]$$

$$\frac{\partial F}{\partial h_j} = \frac{\partial F}{\partial \tau_j} \frac{\partial \tau_j}{\partial h_j}$$

where if the jth asset is the nth to default

$$\frac{\partial F}{\partial \tau_j} = \frac{\partial P}{\partial \tau_j}(\tau_j)[H(T-\tau_j)(1-r_N)]$$

$$- P(\tau_j)[\delta(\tau_j - T)(1-r_n) + H(\tau_j - T) \frac{\partial}{\partial t}(1-r_n)|_{t=\tau_j}]$$

And zero otherwise.
The Pathwise Method for nth Default Swaps

The important terms are the second and third terms.

They correspond to:

\(a.\) default time of \(j\)th asset crosses final maturity of the product.

\(b.\) Upon bumping the \(jth\) hazard rate we alter which asset is the \(nth\) to default

Both result in a jump in value and hence a Delta function in the derivative.
The Pathwise Method for nth Default Swaps

- When differentiated these jumps in the payoff give rise to delta functions!
The Pathwise Method for nth Default Swaps

The delta functions make a bumped Monte Carlo converge very slowly. However, we can integrate these \textit{analytically} to obtain

\[ -P(T) \frac{\partial E^{-1}}{\partial h_j} \int \psi(\tau_1, \ldots, \tau_{j-1}, T, \tau_{j+1}, \ldots, \tau_N) d\tau_1 \ldots d\tau_{j-1} d\tau_{j+1} \ldots d\tau_N. \]

As before we simply reintroduce it, the second term is now

\[ \int \frac{(I\psi(\tau_1, \ldots, \tau_{j-1}, T, \tau_{j+1}, \ldots, \tau_N))}{\psi_{n-1}(\tau_1, \ldots, \tau_{j-1}, \tau_{j+1}, \ldots, \tau_N)} \psi_{n-1}(\tau_1, \ldots, \tau_{j-1}, \tau_{j+1}, \ldots, \tau_N) d\tau_1 \ldots d\tau_{j-1} d\tau_{j+1} \ldots d\tau_N, \]

where \( I = 1 \) if \( t_j \) is the nth default time and zero otherwise.
Delta contributions from recovery rates

Two possible contributions: after sorting $j$th bond becomes $(n-1)$th or $n$th default.

Contribution 1

Contribution 2
Error in convergence of first to default on 4 assets five year deal, 2 percent hazard rates, value of delta 2.015

Seconds

bumped LD
bumped pseudo
bumped antithetic
Error in convergence of first to default on 4 assets five year deal, 2 percent hazard rates, value of delta 2.015
Error in convergence for first to default on 4 assets, five year deal, 2 percent hazard rates, value of delta 2.015
Error in convergence of fourth to default on 4 assets five year deal, 2 percent hazard rates, value of delta 0.01557
Error in convergence for fourth to default on 4 assets, five year deal, 2 percent hazard rates, value of delta 0.01557
General Results

If we run a Monte Carlo simulation for n paths then the standard error is

\[
\frac{\sigma}{\sqrt{n}}
\]

where \( \sigma \) is the standard deviation.

In the following, we therefore plot the standard deviation of the result as a fraction of the result.
standard deviation of delta as a fraction of delta with varying maturity for fourth to default with four assets with varying recovery rates (protection leg only)
standard deviation of delta as a fraction of delta with varying maturity for fourth to default with four assets with varying recovery rates (protection leg only)