# Rapid computation of prices and deltas of nth to default swaps in the Li Model

Mark Joshi,
Dherminder Kainth

QUARC
RBS Group Risk Management

# Summary

- Basic description of an nth to default swap
- Introduction to the Li model
- Solutions: Importance Sampling
- Parameter hedging and why computing sensitivities are difficult.
- Solutions: Likelihood & Pathwise Methods
- Results

# Nth to default swaps: Product definition

- In an nth default swap a regular fee is paid until *n* of a basket of N credits have defaulted, or the deal finishes.

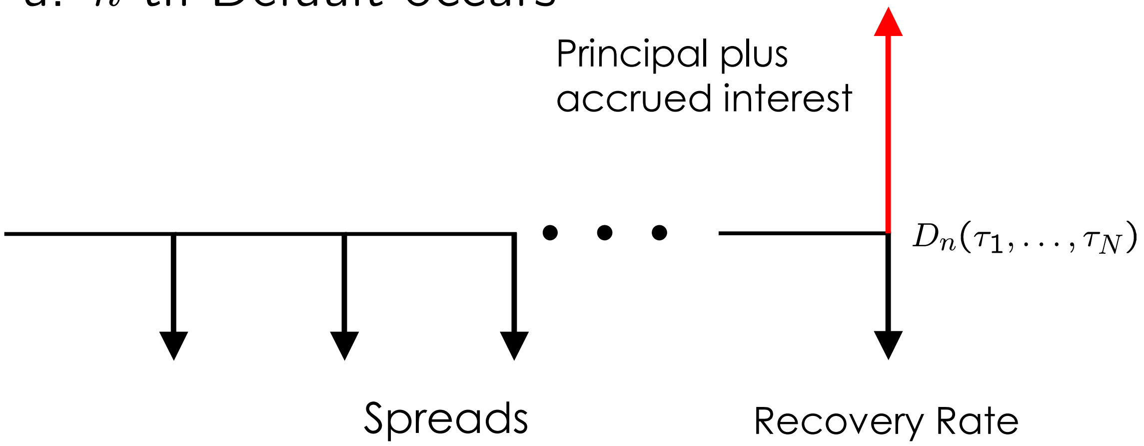
- When the Nth default occurs a payment of

  1 – R is made to the fee payer.

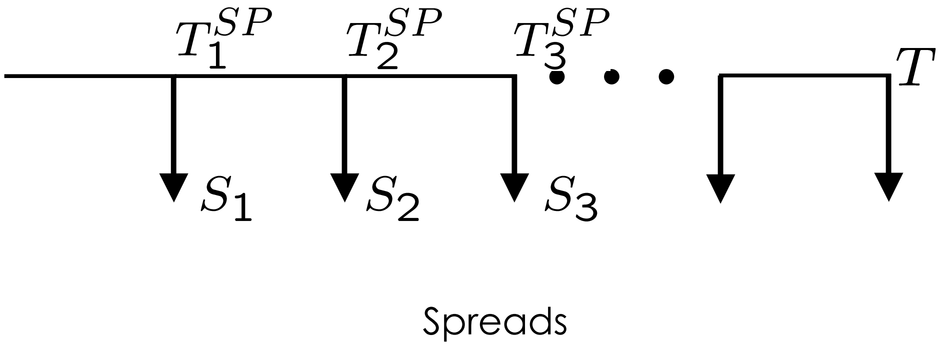
  R = recovery rate of *n*th defaulting asset

# Nth to default swaps: Product definition

a. $n$ th Default occurs



b. $n$ th Default does not occur



# The Li Model

- Defaults are assumed to occur for individual assets according to a Poisson process with a deterministic intensity called the *hazard rate*.

- This means that default times are exponentially distributed.

- Li: Correlate these default times using a Gaussian copula

# Some Definitions

- Consider some security A. We define the default time, $\tau_A$, as the time from today until A defaults.
- We assume the defaults to occur as a Poisson process
- The intensity of this process, h(t), is called the hazard rate.

# The Pricing Algorithm: SetUp

Given a correlation matrix C we compute A such that

$$AA^T = C$$

Let $E(\tau, h)$ denote the cumulative exponential distribution function in $\tau$ given a fixed h:

$$E(\tau, h) = \mathbb{P}(t < \tau) = 1 - \exp(-\int_0^{\tau} h_j(t)dt).$$

$E^{-1}(u, h)$ denotes its inverse for fixed h.

# The Pricing Algorithm

- Draw a vector of independent normals, $\boldsymbol{z}$
- Generate a set of correlated Gaussian deviates:

$$\mathbf{w} = A\mathbf{z}.$$

- Map to uniforms:

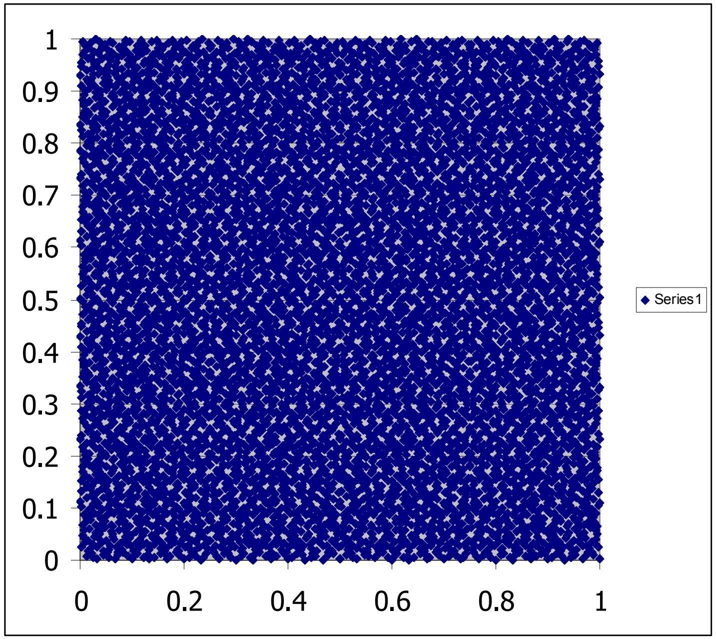
$$u_i = N(w_i)$$

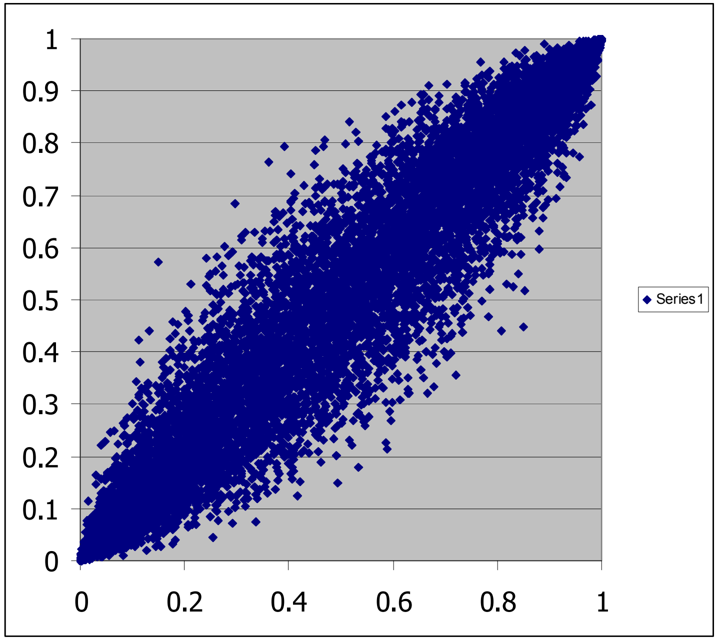
- Map to default times:

$$\tau_i = E^{-1}(u_i, h)$$

- Compute the cash flow in this scenario; discount back.

$$F(\tau_1, \ldots, \tau_N) = P(D_n(\tau_1, \ldots, \tau_N))[V_{\text{prot}} + (1 - r_n)H(T - D_n(\tau_1, \ldots, \tau_N))].$$


1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0
0.2
0.4
0.6
0.8
1
Series1


1
0.9
0.8
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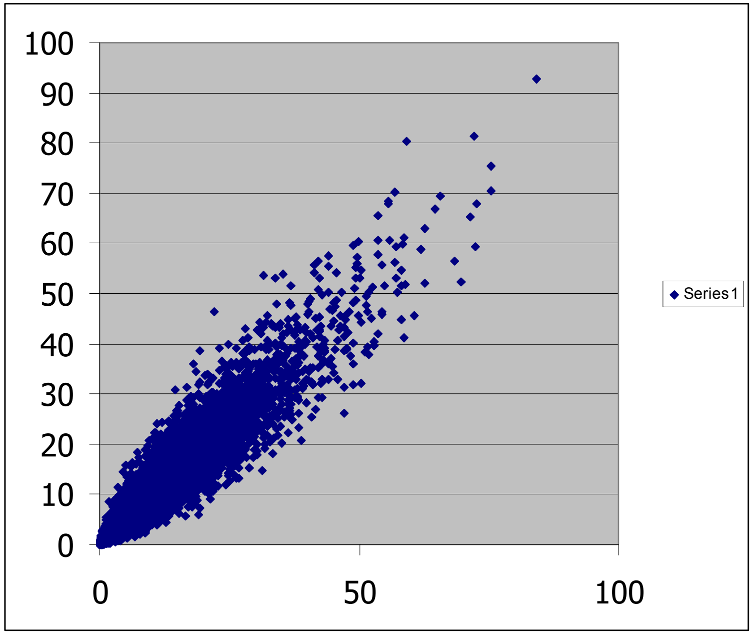
1
Series1


100
90
80
70
60
50
40
30
20
10
0
0
50
100
Series1

# Importance Sampling

Intuitively: want to sample more thoroughly in the regions where defaults occur.

Look at a *k* th to default swap:

- Product pays a constant amount unless *k* defaults occur.
- Restrict our attention to cases of *k* defaults.
- By subtracting the constant, we can assume value is zero unless *k* defaults occur.

# Importance Sampling

- General Strategy: alter the probabilities of default such that we always get *k* defaults. Each path is then"important"; compute prices.

- We then reweight the different contributions according to our changes to the probability measure

# Designing the importance density when *i* =1

Make the *i* th asset default before T with probability:

$$\frac{1}{(n+1)-i}$$

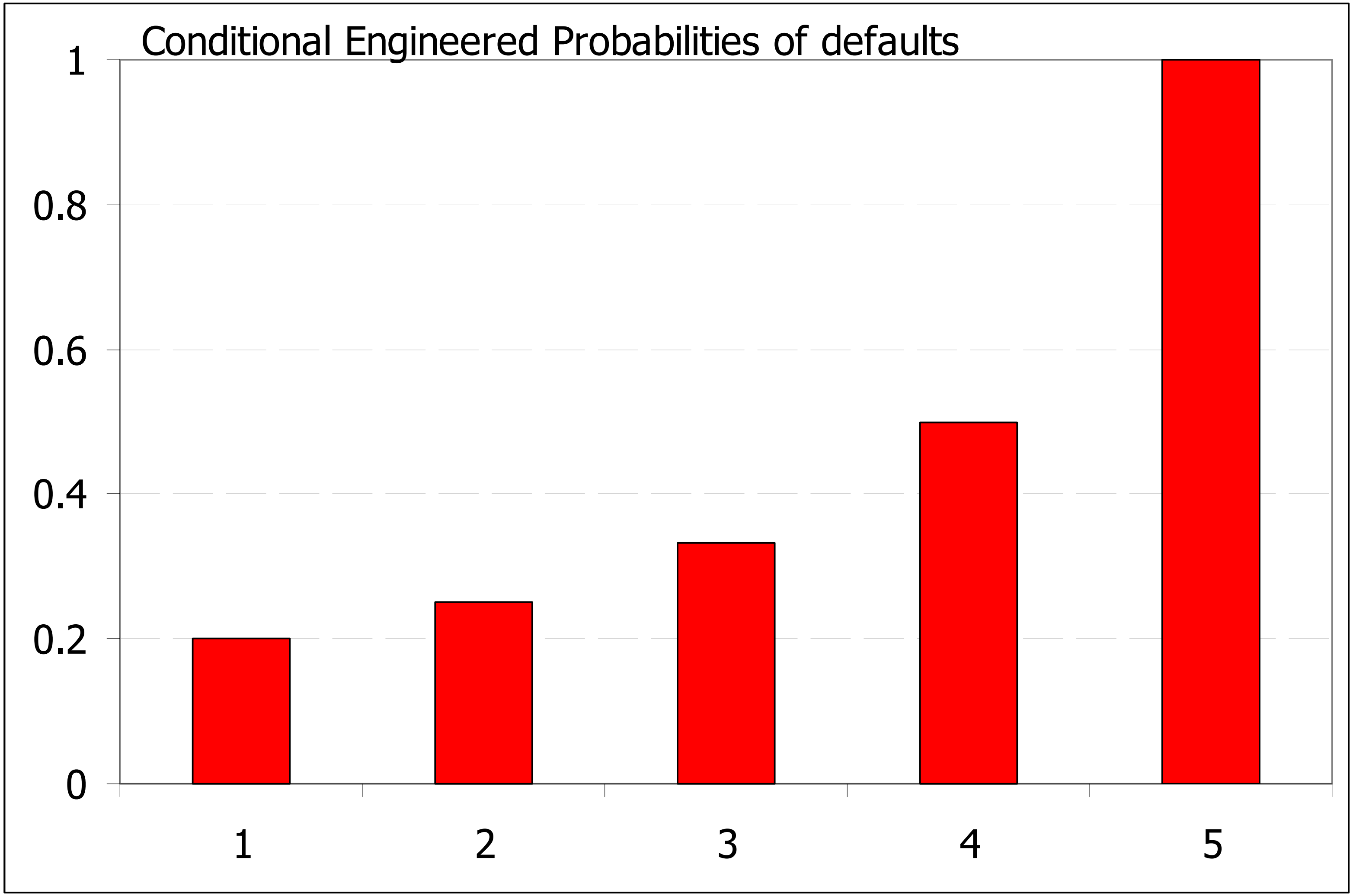
- Why? After *i* non defaults want all the remaining credits to have an equal chance of default

- Pick a uniform $u_i$. If:

$u_i < \dfrac{1}{n+1-i}$ map $u_i$ to a region where asset *i* defaults.

$u_i > \dfrac{1}{n+1-i}$ map $u_i$ a region where asset *i* doesn't default.

# Designing the importance density when $i = 1$



# Designing the importance density

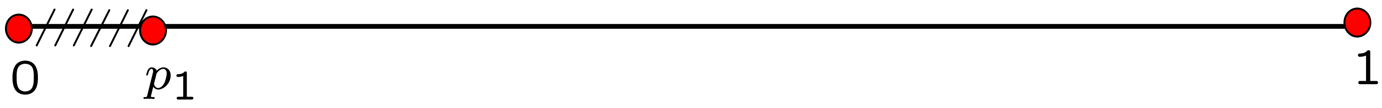
- Look at the original default region for asset *i*

$$\tau_i < T \longrightarrow w_i < x \longrightarrow \sum_j A_{ij} z_j < x$$

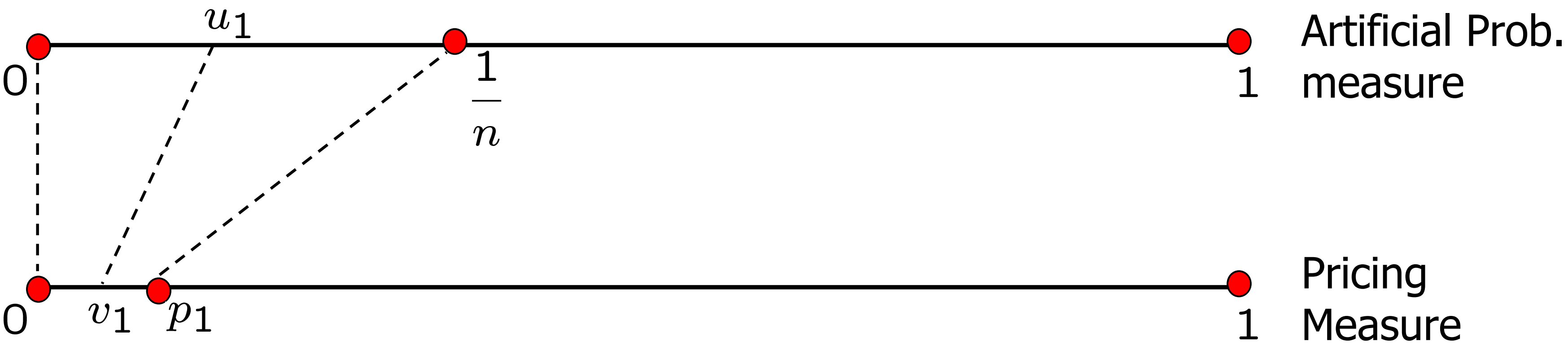
Correlated Gaussian

$N(0,1)$

- For our first to default case: $a_{11} z_1 < x \implies z_1 < \frac{x_1}{a_{11}}$
- Translate to uniforms: $p_1 = N\Big(\frac{x_1}{a_{11}}\Big)$

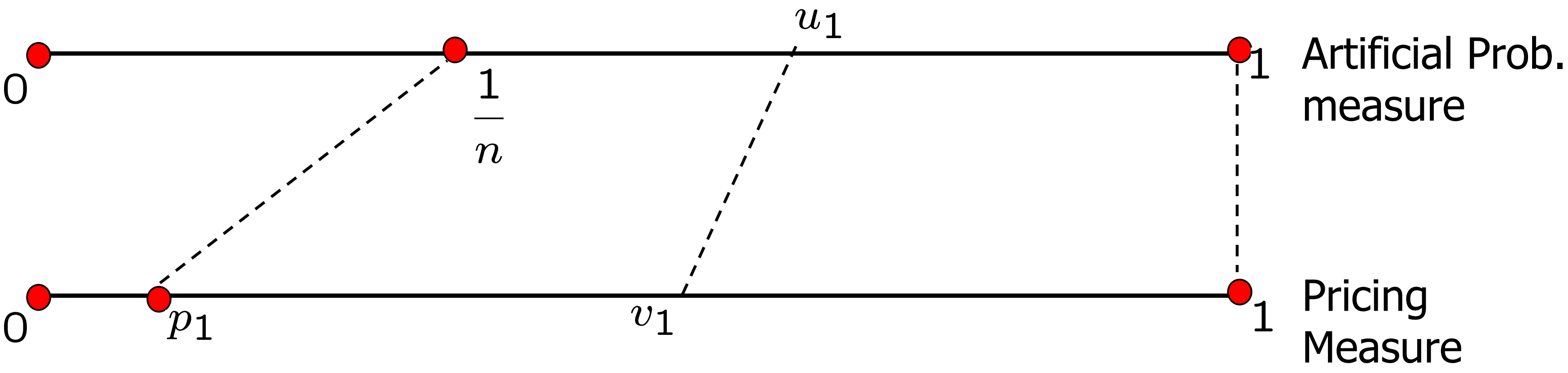


# rst to Default occurs:



- u$_1$ maps to v$_1$ where: $\frac{v_1}{p_1} = u_1 n \implies v_1 = u_1 n p_1$

# rst to Default doesn’t occur:



- u$_1$ maps to v$_1$ where: $v_1 = p_1 + \frac{1-p_1}{1-\frac{1}{n}}(u_1 - \frac{1}{n})$

We need to scale the contributions of these paths

First asset defaults: weight by $np_1$

Doesn't default: weight by $\frac{1-p_1}{1-\frac{1}{n}}$

Suppose that we have dealt with the first (*j* -1) assets. The unmassaged default probability now depends on Z:

$$W_j < x_j \text{ if and only if } \sum_{i<j} a_{ij} Z_i + a_{jj} Z_j < x_j.$$

However, as A is lower triangular we have

$$p_j = \frac{x_j - \sum_{i<j} a_{ij} Z_j}{a_{jj}}$$

And repeat as before.

# Computing Hazard Rate Sensitivities

- We hedge against changes in the hazard rates of the individual assets using "vanilla" default swaps.

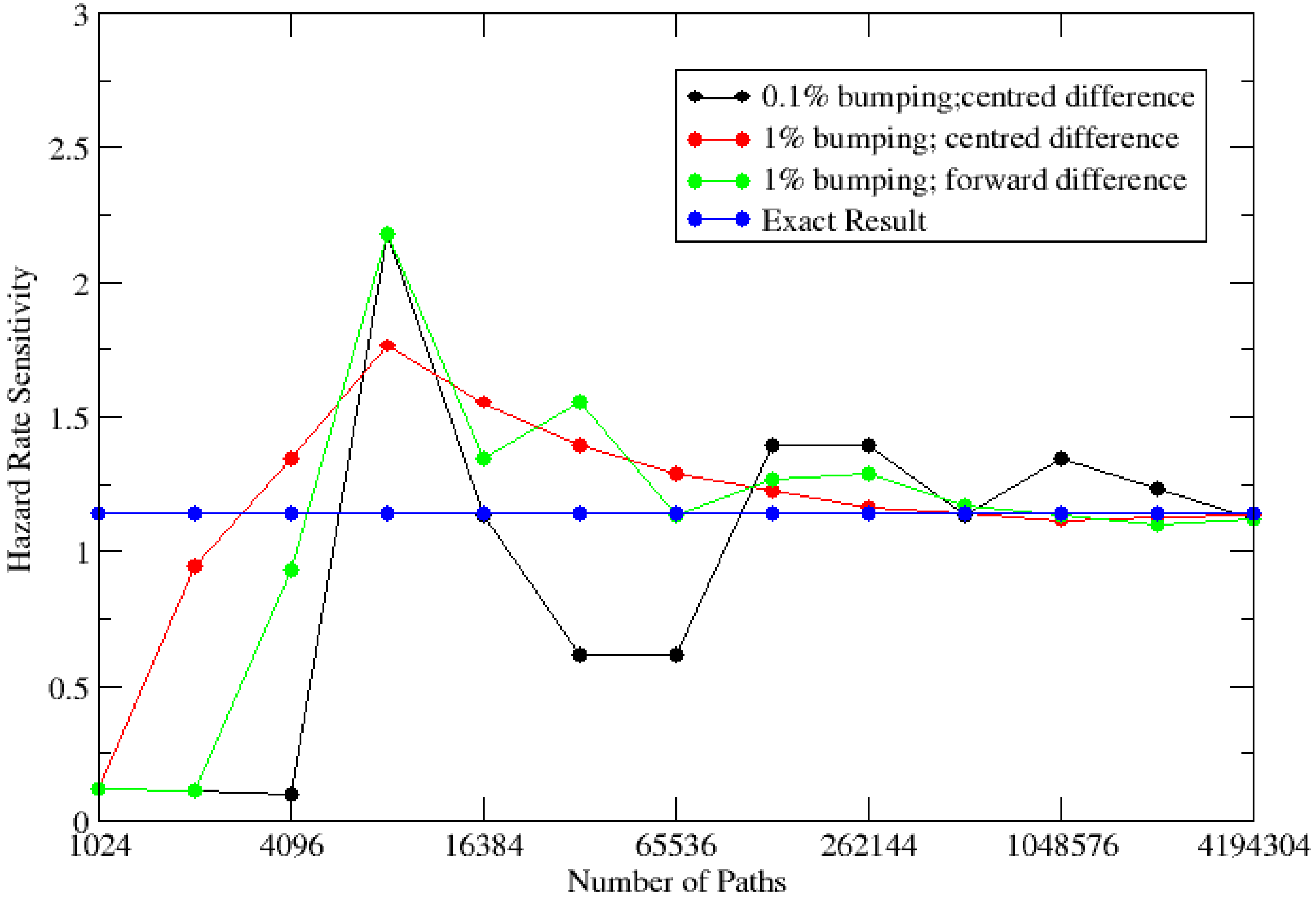
- Naïve methods for determining hazard rate sensitivities (finite differencing)

$$\Delta = \frac{P(h_i + \epsilon) - P(h_i)}{\epsilon} \qquad \text{or} \qquad \Delta = \frac{P(h_i + \epsilon) - P(h_i - \epsilon)}{2\epsilon}$$

have severe limitations due to their (very) slow rate of convergence.

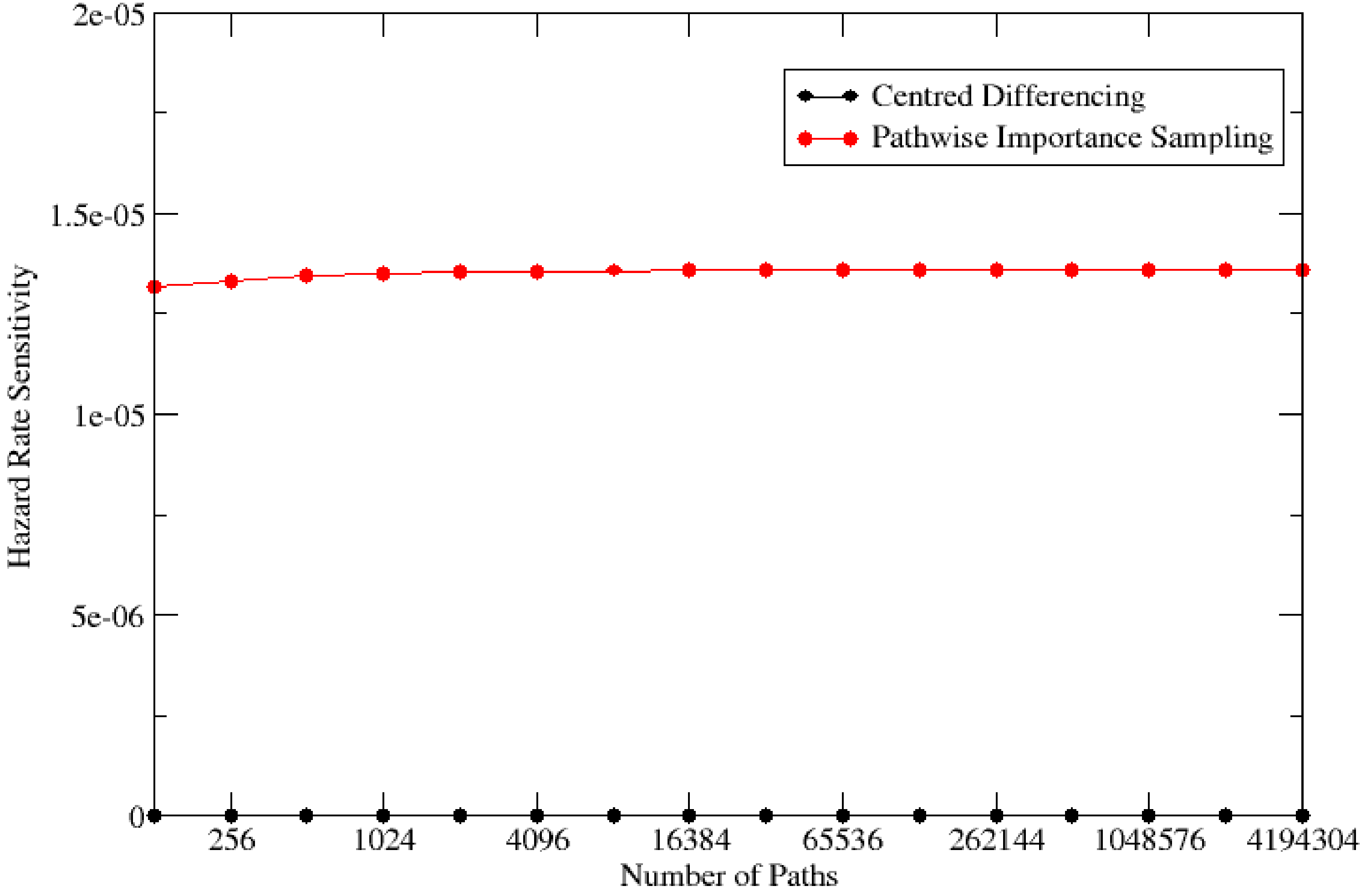
# Computing Hazard Rate Sensitivities

First to default, 4 credits, 2 year deal Not a stress case !



# Computing Hazard Rate Sensitivities

Fourth to default,  4 credits,  0.15 year deal



# Why is Bumping problematic ?

- Very few paths will give multiple defaults a short time (e.g 0.15 years). If obligors are uncorrelated,

$$\text{Prob } n \text{ defaults} = (hT)^n$$

We therefore need lots of paths, even for pricing.

- When we compute sensitivities, bump one hazard rate. Very small change in the number of paths which now have *n* defaults compared to previously.

# Why is Bumping problematic ?

A CDS is similar to a barrier option, pay-out jumps according to whether Nth default is before or after deal maturity.

$$\text{Value CDS} = \int P(D_n(\tau_1,\ldots,\tau_N))[(1-r_n)H(T-D_n(\tau_1,\ldots,\tau_N))\psi(\tau_1,\ldots,\tau_N)]d\tau_1\ldots d\tau_N.$$

When we differentiate the payoff w.r.t the hazard rates we get a δ function.

Sampling this by Monte Carlo is very hard.

# Parameter Sensitivities Using Monte Carlo

Well-known techniques for computing Greeks by Monte Carlo include:

- Likelihood ratio: differentiate the probability density function analytically, inside the integral.

- The Pathwise Method: differentiate the Payoff.
  Generally believed not to apply to discontinuous payoffs – we show that it does apply.
  Broadie-Glasserman

- Malliavin calculus: differentiation w.r.t. the underlying Brownian motion; not applicable here.

# The Likelihood Ratio Method

Value of the option:

$$V = \mathbb{E}^{\mathbb{Q}}[F(S_T)] = \int F(S)\psi(S,\theta)\,dS$$

We can write the sensitivity w.r.t θ:

$$\frac{\partial V}{\partial \theta} = \int F(S)\frac{\partial}{\partial \theta}\psi(S,\theta)\,dS$$

No longer integrating against our Monte Carlo density!
However, we can reintroduce it:

# The Likelihood Ratio Method

$$\frac{\partial V}{\partial \theta} = \int F(S) \frac{\partial \psi(S,\theta)}{\partial \theta} \frac{1}{\psi(S,\theta)} \psi(S,\theta)\, dS$$

$$= \int F(S) \frac{\partial}{\partial \theta} \log \psi(S,\theta) \psi(S,\theta)\, dS$$

$\therefore$ To compute sensitivity we reweight the payoff with:

$$\frac{\partial}{\partial \theta} \log \psi(S,\theta)$$

# The Pathwise Method

- The delta of an option with payoff F($S_T$) is:

$$\Delta = \frac{\partial V}{\partial S_0} = e^{-rT} \int F(S_T) \frac{\partial}{\partial S_0} \psi(S_T, S_0, \ldots)\, dS_T$$

- For the case of a lognormal evolution we can show:

$$\Delta = \frac{\partial \psi}{\partial S_0} = -\frac{\partial}{\partial S_T}\Big(\frac{S_T}{S_0}\psi\Big)$$

- Integrating by parts and eliminating the boundary term:

$$\Delta = e^{-rT} \int \frac{\partial F(S_T)}{\partial S_T} \frac{S_T}{S_0} \psi(S_T, S_0, \ldots)\, dS_T$$

# The Pathwise Method

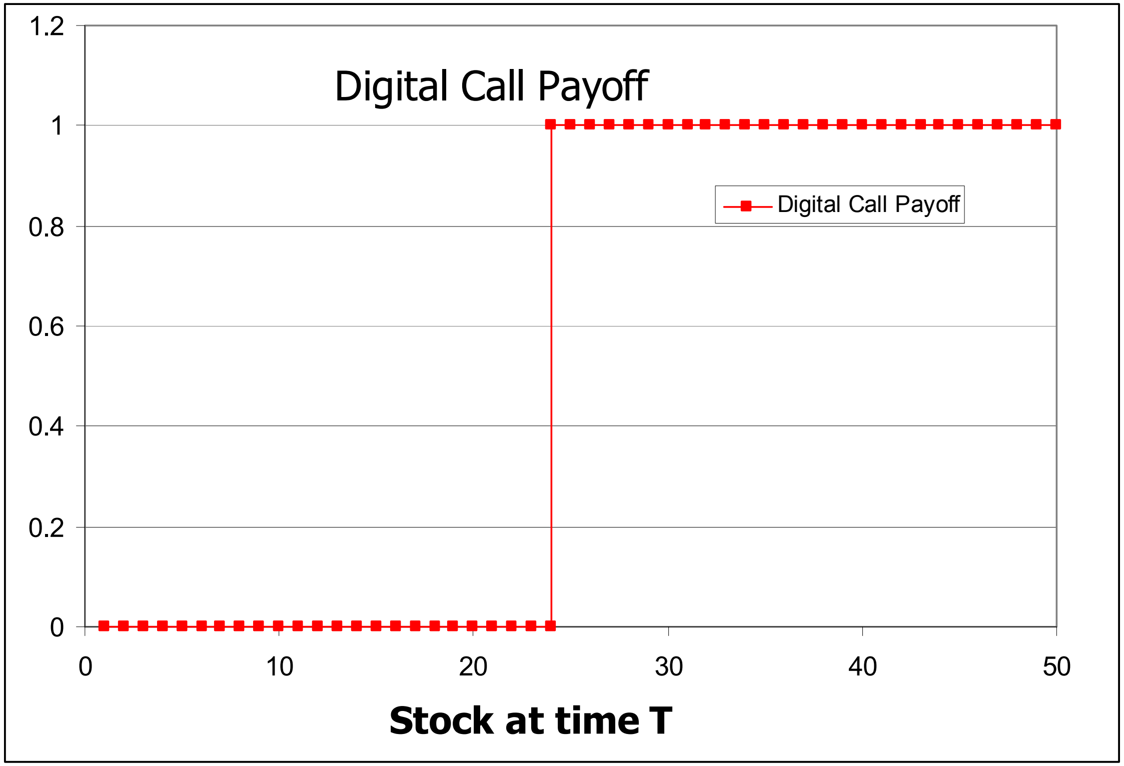
- We are now differentiating the *payoff* !

- Suppose we have a digital option:

$$(S_T) = H(S_T - K)$$

- Differentiate and

we get a δ function



# The Likelihood Ratio Method for nth Default Swaps

- Value of the CDS:

$$\int P(D_n)(1-r_n)H(T-D_n)\psi(\tau_1,\dots,\tau_N)d\tau_1\dots d\tau_N.$$

- Differentiate w.r.t. *i* th hazard rate :

$$\frac{\partial V}{\partial h_i} = \int_0^T P(D_n)(1-r_n)H(T-D_n)\frac{\partial\psi(\tau_1,\dots,\tau_N)}{\partial h_i}d\tau_1\dots d\tau_N.$$

- Applying Broadie/Glasserman's trick:

$$= \int_0^T P(D_n)(1-r_n)H(T-D_n)\frac{\partial \log\psi(\tau_1,\dots,\tau_N)}{\partial h_i}\psi(\tau_1,\dots,\tau_N)d\tau_1\dots d\tau_N.$$

# The Likelihood Ratio Method for nth Default Swaps

- The calculation is straightforward for Gaussian copula and flat hazard rates:

$$\frac{\partial \log \psi(\tau_1, \ldots, \tau_n)}{\partial h_i} = -(\rho^{-1} - 1)_{ij} \eta_j \frac{\partial \eta_i}{\partial u_i} \frac{\partial u_i}{\partial h_i} + \frac{1}{h_i} - \tau_i$$

where ρ is the correlation matrix and

$$\eta_i = \phi^{-1}(u_i) \qquad \frac{\partial \eta_i}{\partial u_i} = \sqrt{2\pi} e^{\frac{1}{2}\phi^{-1}(u_i)^2}$$

# The Pathwise Method for nth Default Swaps

- We differentiate the *discounted pay-off* w.r.t $h_j$ (ignore the spreads for the moment):

$$F(\tau_1,...,\tau_N) = P(D_N(\tau_1,...,\tau_N))[(1-r_n)H(T-D_n(\tau_1,...,\tau_N))]$$

$$\frac{\partial F}{\partial h_j} = \frac{\partial F}{\partial \tau_j}\frac{\partial \tau_j}{\partial h_j}$$

where if the jth asset is the nth to default

$$\frac{\partial F}{\partial \tau_j} = \frac{\partial P}{\partial t}(\tau_j)[H(T-\tau_j)(1-r_N)]$$

$$-P(\tau_j)[\delta(\tau_j - T)(1-r_n) + H(\tau_j - T)\ \frac{\partial}{\partial t}(1-r_n)|_{t=\tau_j}]$$

And zero otherwise.

# The Pathwise Method for nth Default Swaps

The important terms are the second and third terms.

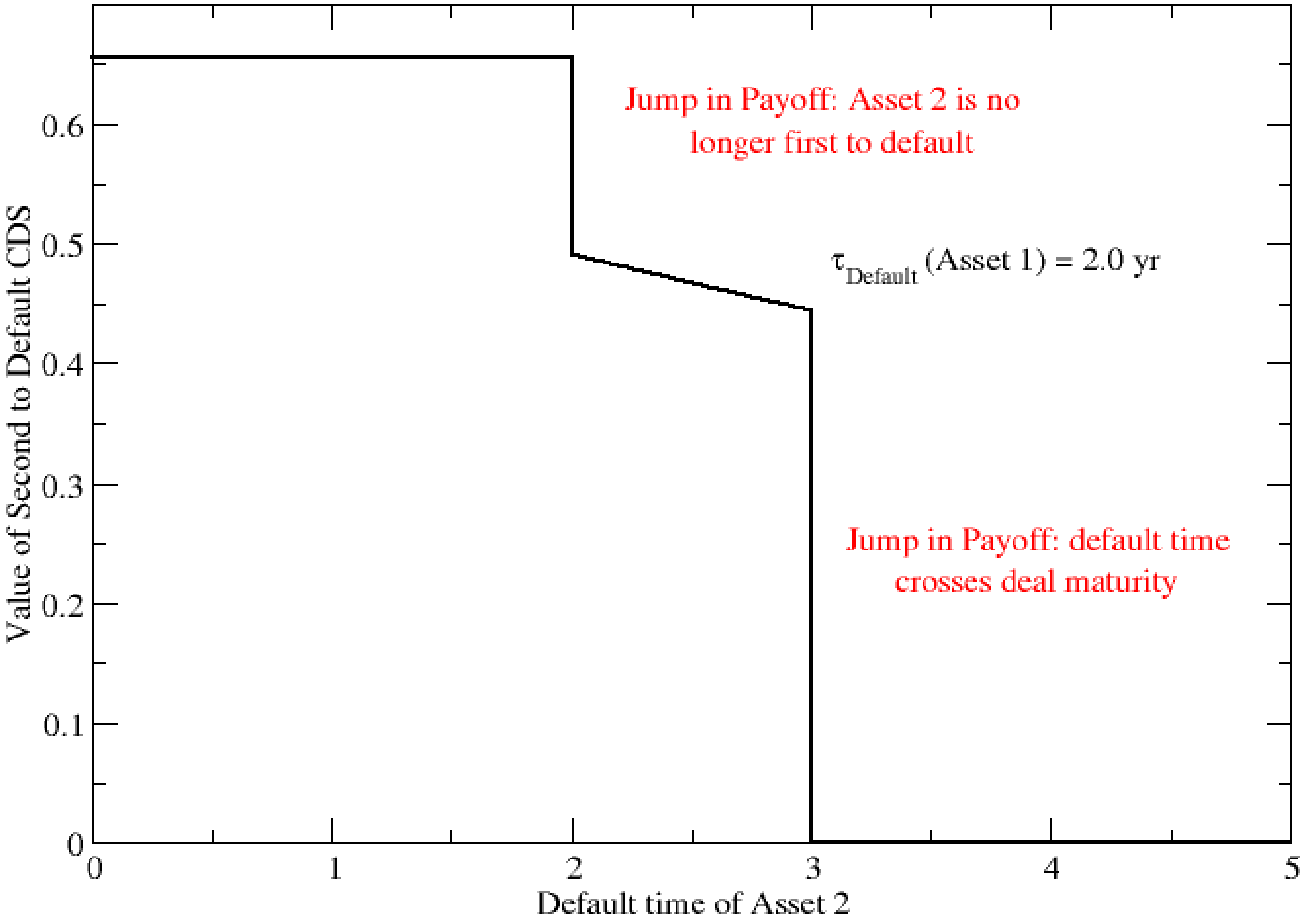
They correspond to:

*a.* default time of $j$ th asset crosses final maturity of the product.

*b.* Upon bumping the *jth* hazard rate we alter which asset is the *nth* to default

Both result in a jump in value and hence a Delta function in the derivative.

# The Pathwise Method for nth Default Swaps



- When differentiated these jumps in the payoff give rise to delta functions !

# The Pathwise Method for nth Default Swaps

The delta functions make a bumped Monte Carlo converge very slowly. However, we can integrate these *analytically* to obtain

$$-P(T)\frac{\partial E^{-1}}{\partial h_j}\int \psi(\tau_1,\dots,\tau_{j-1},T,\tau_{j+1},\dots,\tau_N)d\tau_1\dots d\tau_{j-1}d\tau_{j+1}\dots d\tau_n.$$

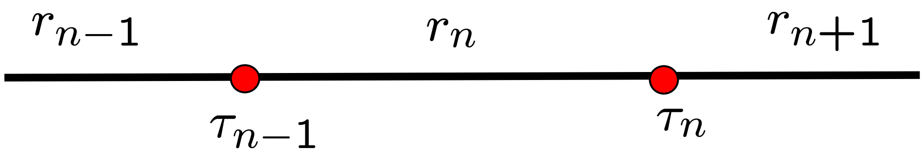
As before we simply reintroduce it, the second term is now

$$\int \frac{(I\psi(\tau_1,\dots,\tau_{j-1},T,\tau_{j+1},\dots,\tau_N))}{\psi_{n-1}(\tau_1,\dots,\tau_{j-1},\tau_{j+1},\dots,\tau_N)}$$
$$\psi_{n-1}(\tau_1,\dots,\tau_{j-1},\tau_{j+1},\dots,\tau_N)d\tau_1\dots d\tau_{j-1}d\tau_{j+1}\dots d\tau_n,$$

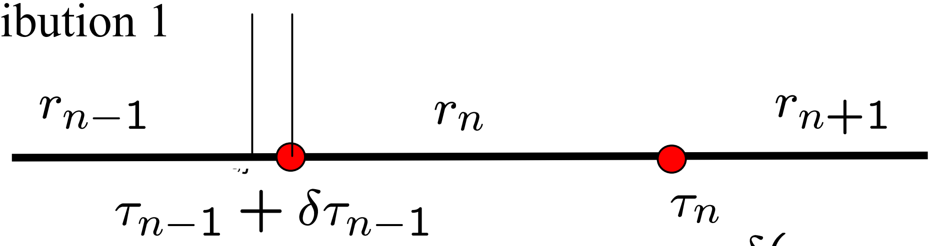
where I = 1 if $t_j$ is the nth default time and zero otherwise.

# Delta contributions from recovery rates

Two possible contributions: after sorting *j* th bond becomes (*n-1*)th or *n* th default.

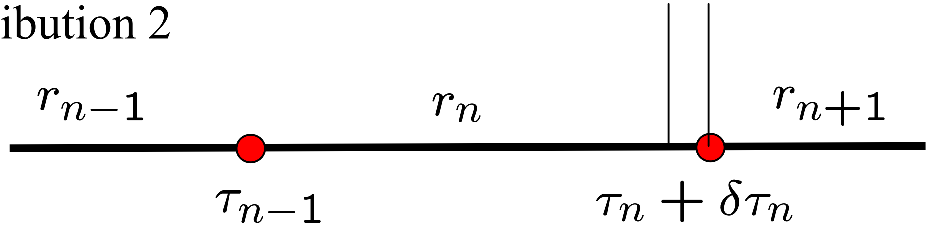


ontribution 1

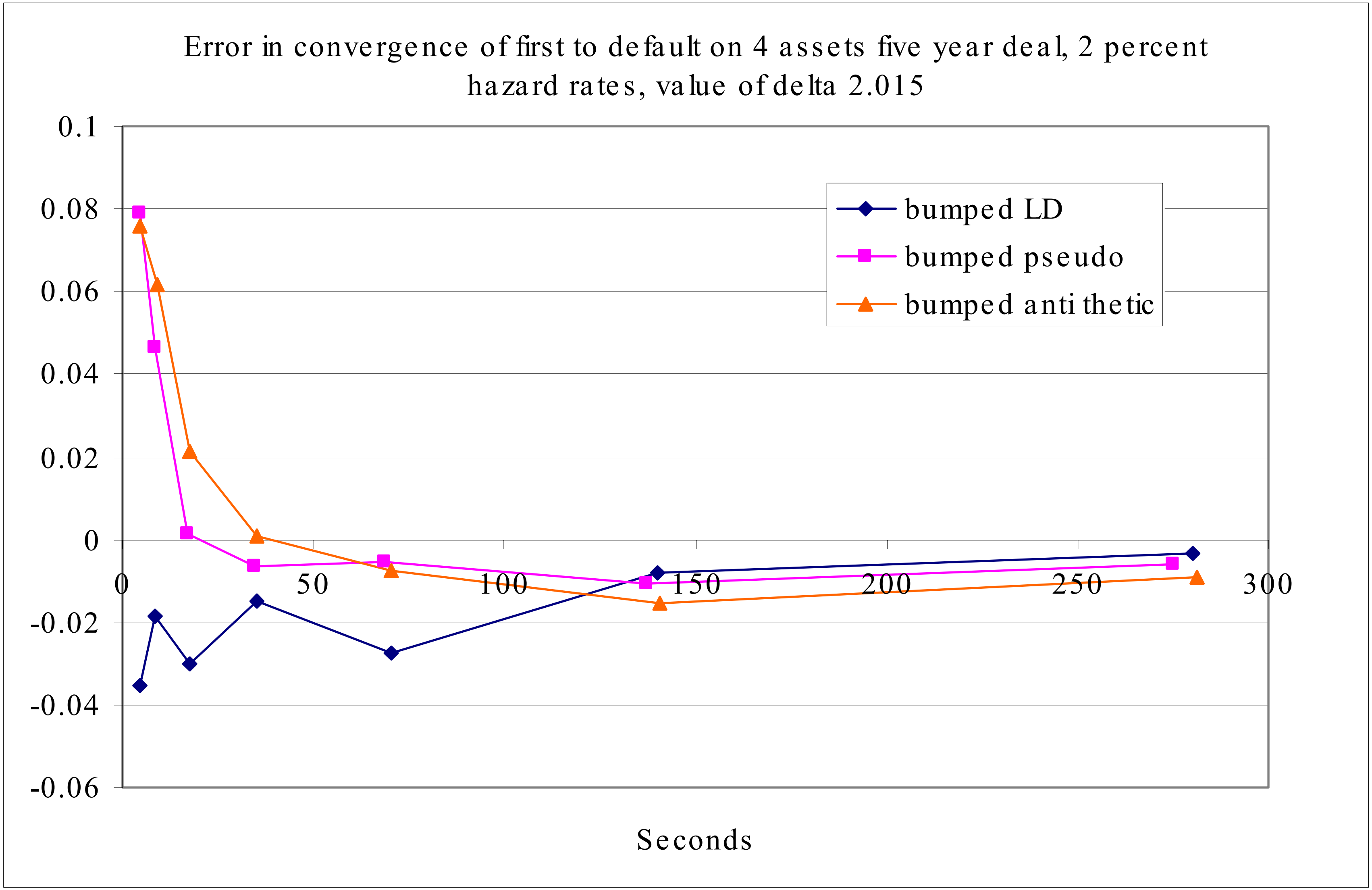


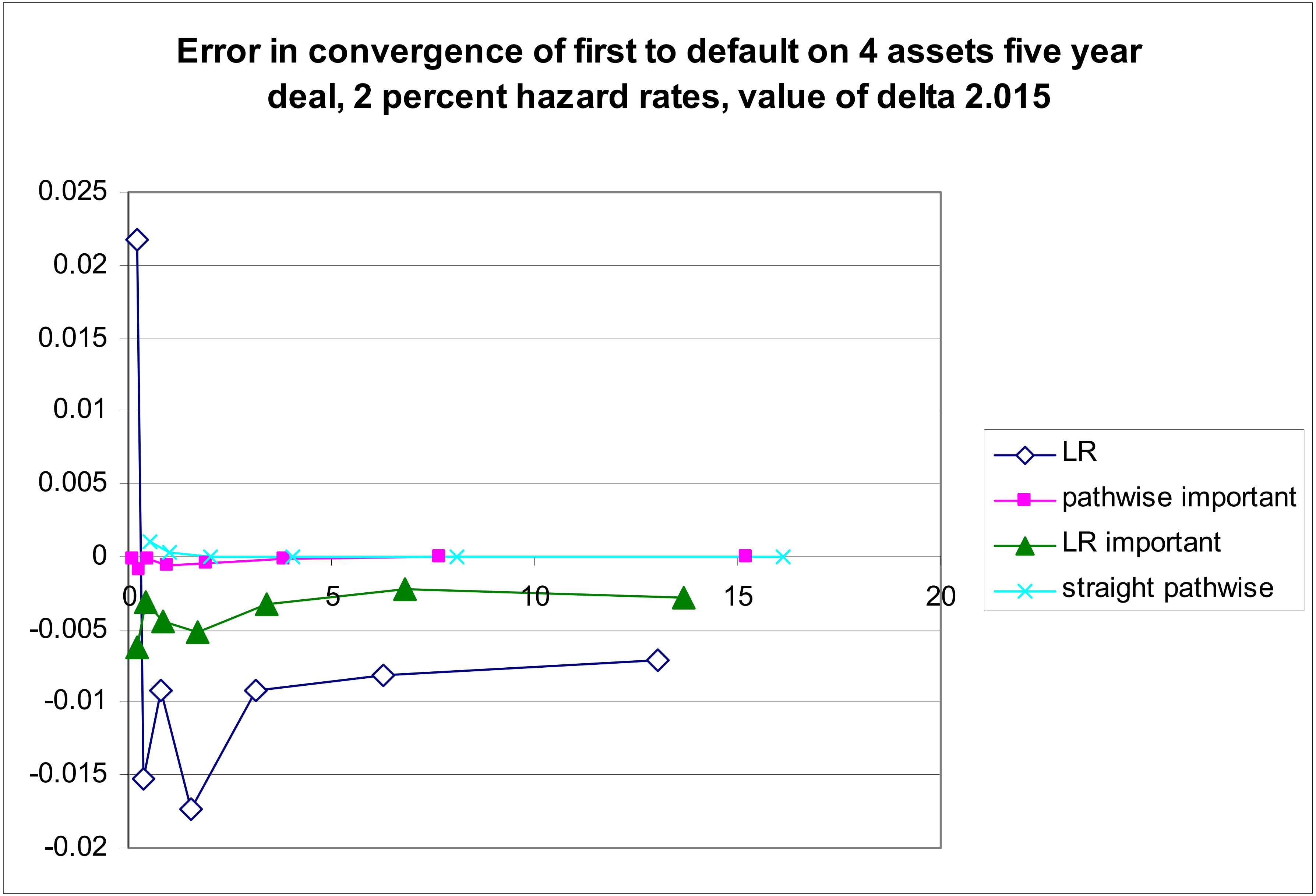
$$\delta(\tau_{n-1} - T)[((1 - r_n) - (1 - r_{n-1}))P(T)]$$

ontribution 2

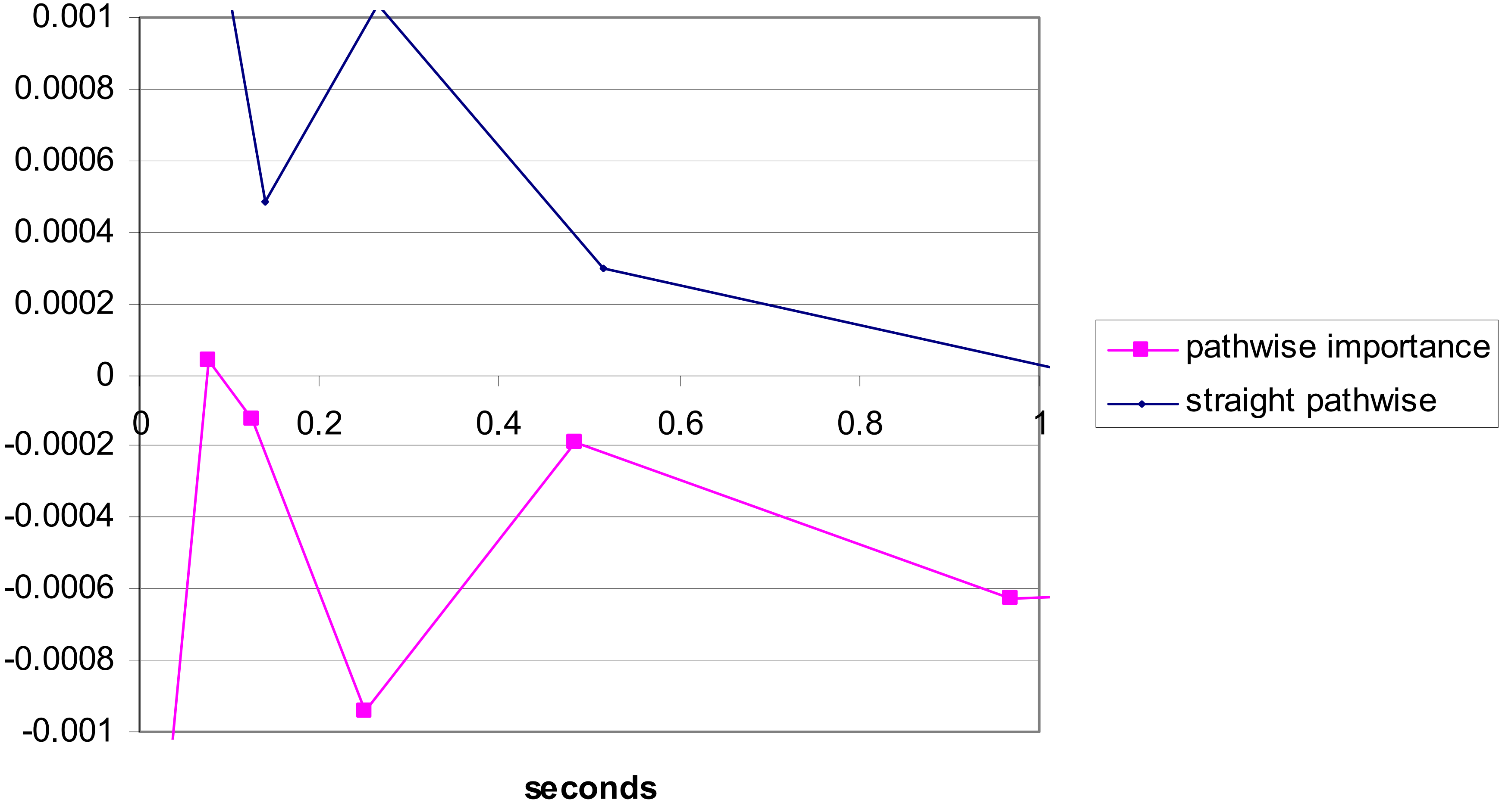


$$\delta(\tau_n - T)[((1 - r_{n+1}) - (1 - r_n))P(T)]$$

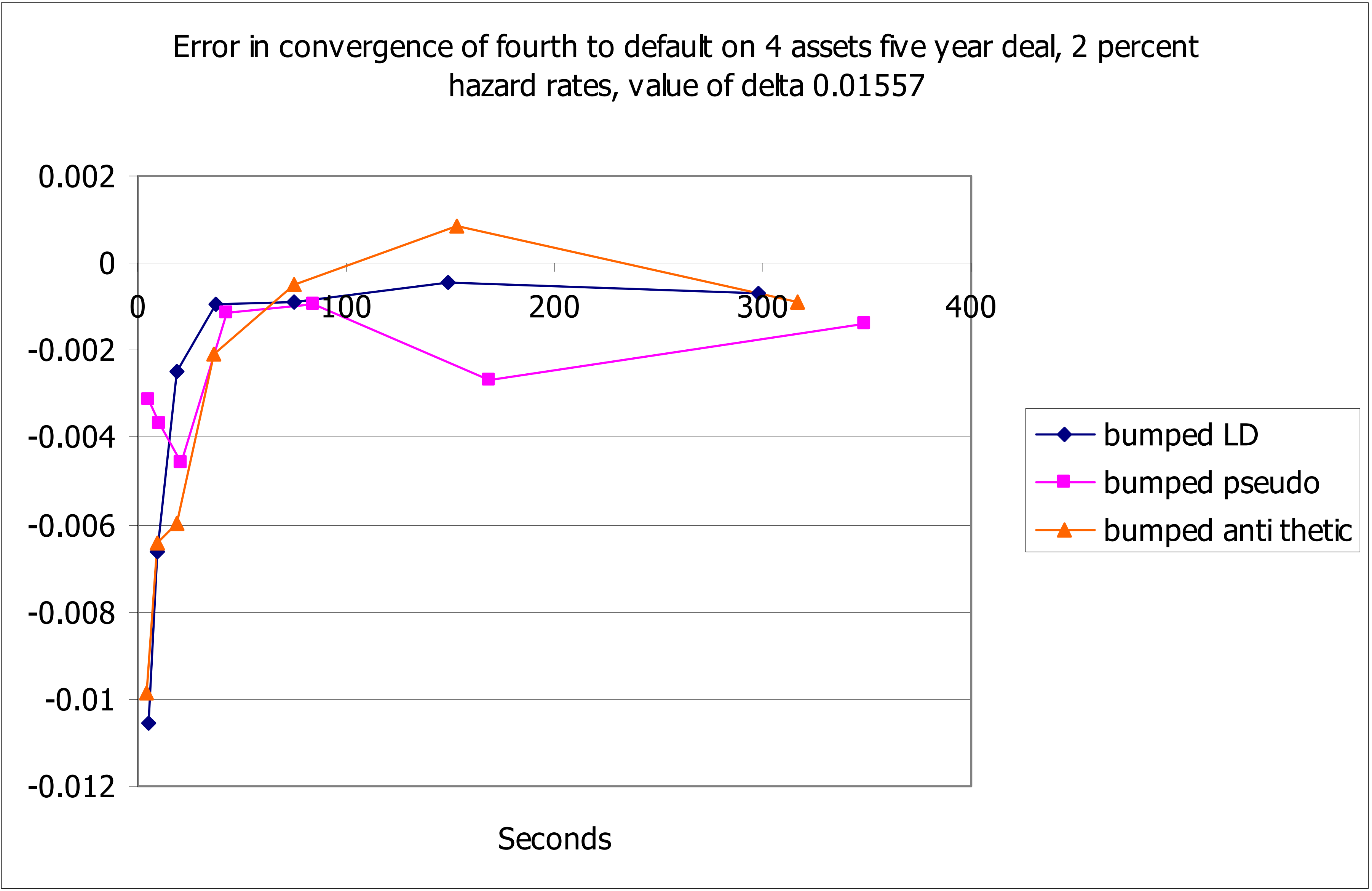

Error in convergence of first to default on 4 assets five year deal, 2 percent hazard rates, value of delta 2.015
0.1
0.08
0.06
0.04
0.02
0
-0.02
-0.04
-0.06
0
50
100
150
200
250
300
Seconds
bumped LD
bumped pseudo
bumped anti thetic



Error in convergence of first to default on 4 assets five year deal, 2 percent hazard rates, value of delta 2.015

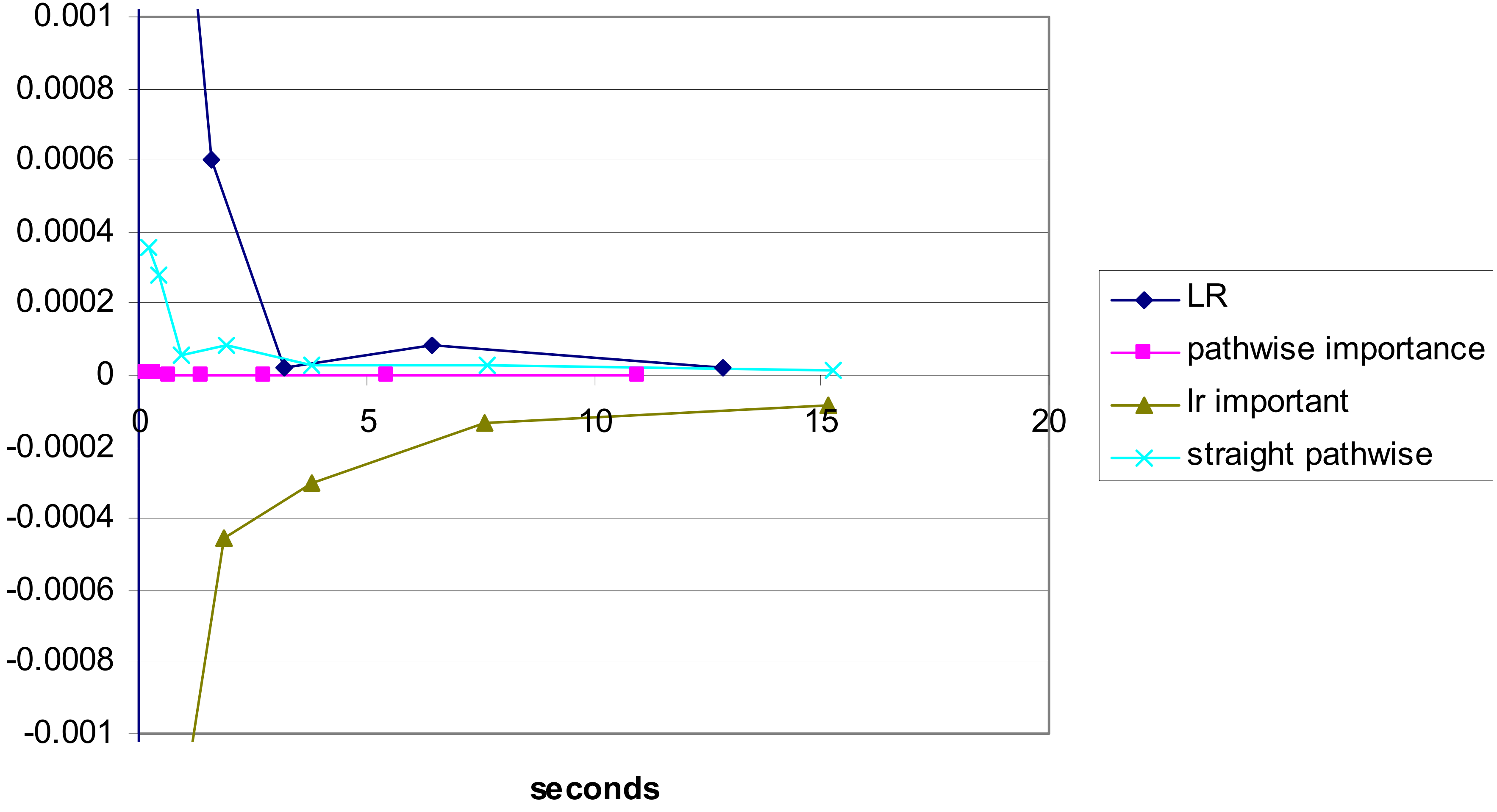


Error in convergence for first to default on 4 assets, five year deal, 2 percent hazard rates, value of delta 2.015



Error in convergence of fourth to default on 4 assets five year deal, 2 percent hazard rates, value of delta 0.01557

Error in convergence for fourth to default on 4 assets, five year deal, 2 percent hazard rates, value of delta 0.01557



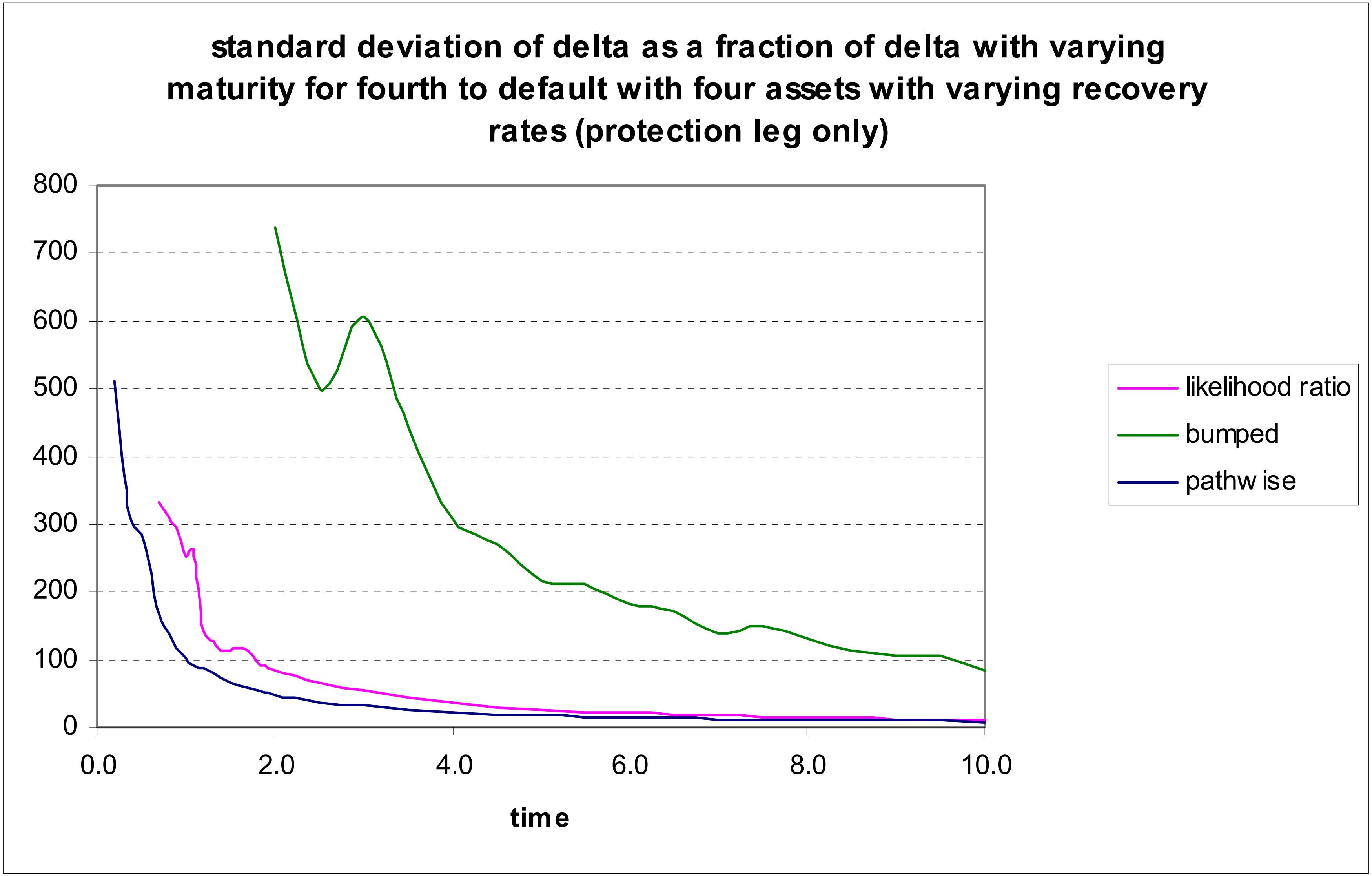
# General Results

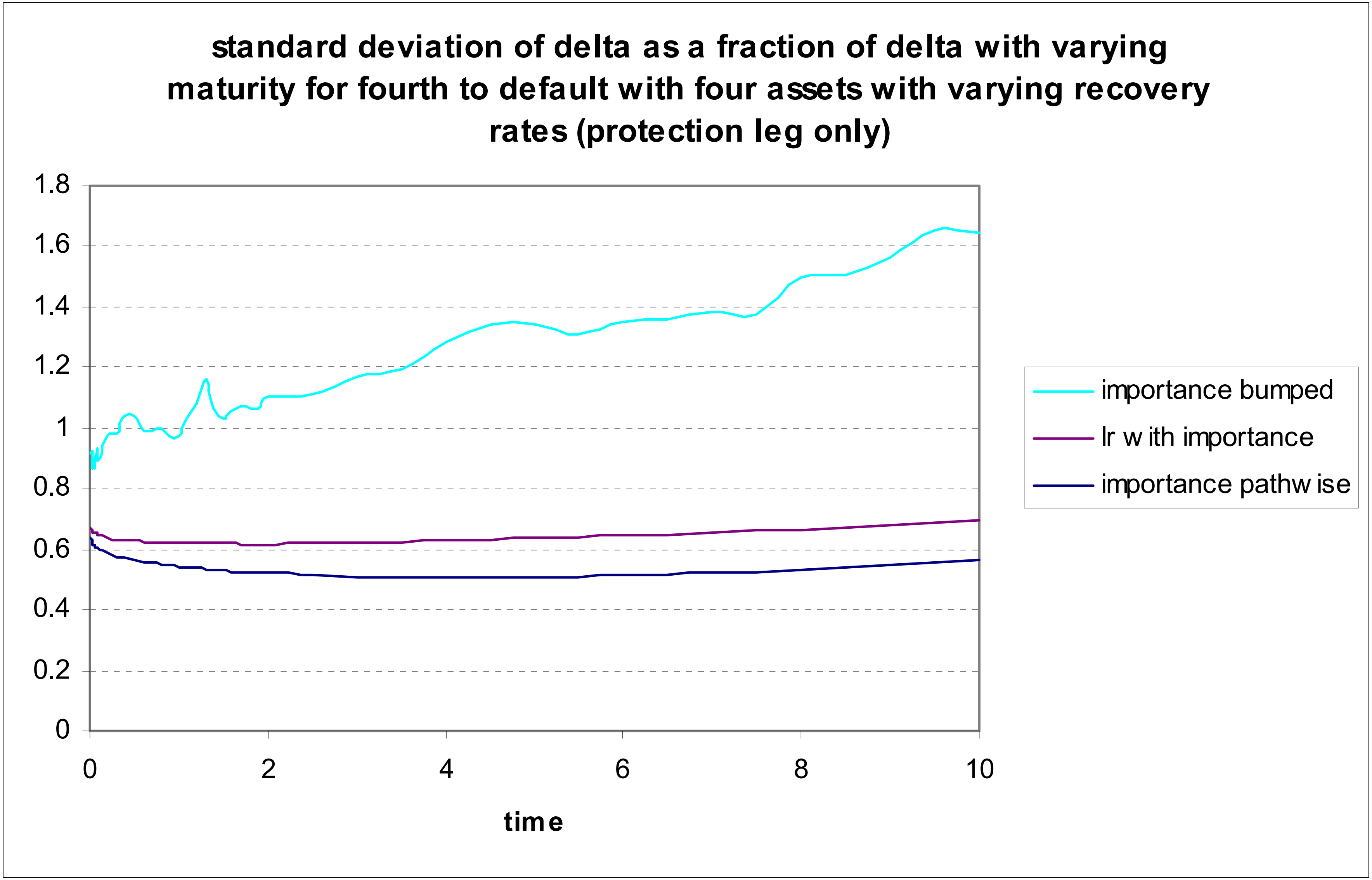
If we run a Monte Carlo simulation for n paths then the standard error is

$$\frac{\sigma}{\sqrt{n}}$$

where $\sigma$ is the standard deviation.

In the following, we therefore plot the standard deviation of the result as a fraction of the result.


standard deviation of delta as a fraction of delta with varying maturity for fourth to default with four assets with varying recovery rates (protection leg only)
800
700
600
500
400
300
200
100
0
0.0
2.0
4.0
6.0
8.0
10.0
time
likelihood ratio
bumped
pathwise



standard deviation of delta as a fraction of delta with varying maturity for fourth to default with four assets with varying recovery rates (protection leg only)