LOG-TYPE MODELS, HOMOGENEITY OF OPTION PRICES AND CONVEXITY

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Abstract. It is shown that the properties of convexity of call prices with respect to spot price and homogeneity of call prices as a function of spot and strike hold for a large class of models of stock price evolution. It is also shown that these properties hold for wider classes of options. A converse result is also demonstrated.

1. Introduction

Many standard models of asset price evolution share the property that in the martingale measure increments of the log of the asset price are independent of the current value of the asset price. This property holds in particular for the Black-Scholes model, Merton's jump-diffusion model, [12], the Variance Gamma model, [11, 10, 9], and for many stochastic volatility models for example [4, 6, 7, 8]. We shall call such models log-type. Simple examples of models which are not log-type are the Bachelier model, [1], and the Dupire model, [5]. Our purpose in this note is to demonstrate that certain nice properties of the Black-Scholes price of a call option actually hold in any log-type model for a call option. In particular, we show that the call price is still a homogeneous function of spot and strike and that its Gamma (i.e. the second derivative with respect to spot) is always non-negative. We also show extend these results to more general classes of pay-offs and prove an inverse result that if call prices are homogeneous, then the model is of log-type.

The definition of log-type is equivalent to the statement that the density function for the evolution of the log between two times is expressible as a difference

$$\Phi_{t,T}(\log S_T - \log S_t) d(\log S_t),$$

or equivalently the density for the evolution of the spot is of the form

$$\Phi(S_T/S_t) dS_T/S_T.$$

Note that Φ will be supported purely on the positive real axis, and throughout we asssume that our asset prices are strictly positive.

2. Homogeneity

Let $C(S_0, K, T)$ denote the price of a call option struck at K, with expiry T at time 0 if spot is S_0. Our first result is

Theorem 2.1. *The function $C(S_0, K, T)$ is homogeneous of order 1 in (S_0, K). i.e.*

$$C(\lambda S_0, \lambda K, T) = \lambda C(S_0, K, T).$$

Proof. We have, dropping T, that

$$C(S_0, K) = \int (S - K)_+ \Phi\left(\frac{S}{S_0}\right) \frac{dS}{S} \tag{1}$$

$$C(\lambda S_0, \lambda K) = \int (S - \lambda K)_+ \Phi\left(\frac{S}{\lambda S_0}\right) \frac{dS}{S} \tag{2}$$

$$= \int (\lambda S' - \lambda K)_+ \Phi\left(\frac{\lambda S'}{\lambda S_0}\right) \frac{dS'}{S'} \tag{3}$$

$$= \lambda \int (S' - K)_+ \Phi\left(\frac{S'}{S_0}\right) \frac{dS'}{S'} \tag{4}$$

$$= \lambda C(S_0, K). \tag{5}$$

Here in the second equality we have performed a change of variables $S = \lambda S'$. □

Note that this proof will hold for any pay-off function which is a homogeneous function of spot and strike. We therefore have

Corollary 2.1. *If the derivative, $D(S, K, T)$, has pay-off which is a homogeneous of degree one function of spot and strike then the price is a homogeneous degree one function of spot and strike for any log-type model.*

3. Convexity

We can also show that the Gamma of a call option or indeed any option with a convex pay-off is positive for any log-type model. Recall that a function is convex if the chord between any two points on the graph lies above the graph, and that convexity is equivalent to the fact that the second derivative is non-negative.

When a function such is not twice differentiable, we can interpret this result in a distribution sense. For example, the call option's pay-off has second derivative equal to $\delta(S-K)$. The distribution $\delta(S-K)$ is positive as if its pairing with any positive function is positive as if $f(x) > 0$ for all x, we have

$$\int f(x)\delta(S-K)dS = f(K) > 0.$$

Theorem 3.1. *If the derivative D has pay-off, f, at time T which is a convex function of S_T then the Gamma of D in any log-type model is non-negative, provided we have that $f(S_T)\phi(S_T/S_0)$ tends to zero as S_T tends to infinity.*

Note that the technical condition here is very mild.

Proof. We have that the value of D at time zero is equal to

$$e^{-rT}\int f(S)\phi\left(\frac{S}{S_0}\right)\frac{dS}{S}.$$

Differentiating with respect to S_0 we obtain

$$-e^{-rT}\int f(S)\phi'\left(\frac{S}{S_0}\right)\frac{dS}{S_0^2}.$$

Integrating by parts and using the technical part of the hypothese, this becomes

$$e^{-rT}\int f'(S)\phi\left(\frac{S}{S_0}\right)\frac{dS}{S_0}.$$

We now change variables and letting

$$\bar{S} = \frac{S}{S_0},$$

we get

$$e^{-rT}\int f'(\bar{S}S_0)\phi\left(\bar{S}\right)d\bar{S}.$$

Differentiating with respect to S_0, we get

$$e^{-rT}\int f''(\bar{S}S_0)\bar{S}\phi\left(\bar{S}\right)d\bar{S}.$$

This will be non-negative as f is convex and ϕ is supported where $\bar{S}$ is non-negative. As the Gamma is non-negative the value is convex as a function of spot. □

4. A converse

Our final result is a converse. We rely on the result of Breeden and Litzenberger, [3], that the risk-neutral density is the second derivative of the call price with respect to strike, divided by the value of a zero-coupon bond with the sam expiry.

Theorem 4.1. *If call option prices are homogeneous of degree one then the risk-neutral density function of the spot at time T is of the form*

$$\Psi(S/S_0)dS/S.$$

Proof. Let $\Phi(S, S_0)$ be the density. The homogeneity of call prices is equivalent to the statement $\lambda^{-1}C(\lambda S, \lambda S_0)$ is independent of λ. So we have that

$$\int (\lambda^{-1}S - K)_+ \Phi(S, \lambda S_0) dS,$$

is independent of S. Letting $S = \lambda S'$, we have

$$\int \lambda(S' - K)_+ \Phi(\lambda S', \lambda S_0) dS'.$$

Differentiating twice with respect to K and setting $K = S$, we have that

$$\lambda \Phi(\lambda S, \lambda S_0)$$

is independent of λ. Putting λ equal to one and $1/S$, this implies that

$$\Phi(S, S_0) = S^{-1}\Phi(1, S^{-1}S_0).$$

Putting

$$\Psi(x) = \Phi(1, x^{-1}),$$

the result follows. □

References

[1] L. Bachelier, Theorie de la Speculation, Annales Scientifiques de L'Ecole Normale Superieure III-17, 21-86 (1900)

[2] F. Black, M. Scholes, The Pricing of Options and Corporate Liabilities, *Journal of Political Economy*, 81, 637-654

[3] D. Breeden, R.Litzenberger, Prices of state-contingent claims implicit in option prices. *J. Business* 51, 621-651

[4] J-P Fouque, G. Papanicolaou, K. R. Sircar, Derivatives in Financial Markets with Stochastic Volatility, Cambridge University Press 2000

[5] B. Dupire, Pricing with a smile, *Risk*, 1994

[6] S. Heston, A closed-form solution for options with stochastic volatility with applications to bond and currency options. *Review of Financial Studies* **6(2)**, 327-343

[7] J. Hull, A. White, The pricing of options on assets with stochastic volatilities, Journal of Finance, **2**, 281-300

[8] A.L. Lewis, Option Valuation Under Stochastic Volatility, Finance Press 2000, Newport Beach, California, USA

[9] D. Madan and E. Seneta, The Variance Gamma model for share market returns, J. Business 63, 511–524, 1990.

[10] D. Madan and F. Milne. Option Pricing with V.G. Martingale Components, Mathematical Finance, 1, 4, 39-55, 1991.

[11] D. Madan, P. Carr and E. C. Chang, The Variance Gamma Process and Option Pricing European Finance Review, Vol. 2, No. 1, 1998

[12] R. Merton, Option pricing when underlying stock price returns are discontinuous. *J. Finan. Econom.* 8, 125-144

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