Accurate and optimal calibration to co-terminal European swaptions in a FRA-based BGM framework

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1 – Introduction and motivation

One of the most pressing problems in interest-rate derivatives pricing is the evaluation of Bermudan swaptions. This product presents greater challenges than path-dependent derivatives in a BGM log-normal-forward-rate framework because of the requirement to estimate the exercise free boundary. Several techniques have recently been proposed, however, that provide accurate solutions using 'smart' Monte Carlo techniques, (which, unlike trees, are best suited to a BGM environment). The outstanding problem is therefore the calibration of the BGM model to be fed into the Monte Carlo engine. This problem has been tackled along two distinct lines: i) by re-casting the BGM formalism in terms of the continuous-time evolution of swap rates, as pioneered by Jamshidian (1997); ii) by attempting to recover the prices of the co-terminal European swaptions from the dynamics of the forward rates.

Route i) is conceptually appealing, but suffers from numerical difficulties: the expressions for the drift corrections of swap rates are considerably more daunting and time-consuming to evaluate than the corresponding quantities for forward rates[1]; in one formulation, they require the knowledge of rather opaque quantities, such as, e.g., the forward rate/swap rate correlation; most importantly, they take as inputs the instantaneous volatilities of forward rates that are implied by the posited swap rate dynamics, and which are therefore not *a priori* known to the user.

Route ii) is much easier to implement from the numerical point of view, but leaves unanswered the question of how the dynamics of the forward rates should be specified in such a way that the prices of the important co-terminal European swaptions are correctly recovered.

[1] Jamshidian (1997) and Rebonato (1999) provide different but equivalent expressions for the drift correction that depend on different market-related volatilities and correlations.

This paper makes a contribution in this direction by providing a calibration methodology that recovers (almost) exactly the prices of all the co-terminal swaptions underlying a given Bermudan swaption, and, at the same time, ensures optimal recovery of user-specified portions of the forward-rate covariance matrix (for instance, the attention could be focussed on caplet prices). The ability to obtain at the same time a virtually perfect fit to European swaptions and an optimal calibration to a user-specified forward-rate covariance matrix is extremely important, since virtually no exotic product depends on the dynamics of one set of state variables only (forward rates or swap rates). See, e.g., Sidenius (2000) or Jamshidian (1997).

2 -Calibration to co-terminal European swaption prices

The result mentioned above are achieved by combining and extending the suggestions provided in Rebonato (1999a) and Rebonato and Jaeckel (2000b). The first paper shows how a FRA-based BGM model can be effectively calibrated to a correlation matrix using an unconstrained optimization; the second presents an efficient and accurate approximation to obtain semi-analytically the price of a European swaption, given the instantaneous volatilities and correlations of the underlying forward rates.

More precisely, the main result obtained in Rebonato and Jaeckel (2000b) is that one is justified in assuming that, as long as one is interested in the price of plain-vanilla European swaptions, some suitably defined weights $\zeta_{jk}(t)$ that enter the expression of the swap rate volatility can be effectively approximated by assuming that they are the same as would be calculated using today's yield curve. This result is used in this section in order to achieve a calibration of a forward-rate-based implementation to the co-terminal European swaption prices underlying a given Bermudan swaption in such a way that the forward rate dynamics is as close as possible to a desired ('target') one. As I shall show below, the exercise is not trivial, since a variety of procedures[2] can give rise to the same exact fitting to the European swaptions. Each procedure, however, will in general produce significantly different Bermudan swaption prices. I will argue that, if the final

[2] I will show that actually the fit to European swaption prices can be achieved in an *infinity* of ways.

purpose is to price a Bermudan swaption, a good criterion to choose among the various possible parametrizations is to require that all the European swaptions should be exactly priced and, *at the same time*, plausible values should be obtained for the caplet prices and/or for the forward-rate/forward-rate correlation matrix.

The setting is the standard BGM environment, described, e.g. in Brace, gatarek and Musiela (1995), Musiela and Rutkowski (1997) or Jamshidian (1997), and therefore will not be repeated here. We shall have all the generality required, and more, by requiring that all the volatilities presented in the following should be L^2-integrable deterministic function of time. Before embarking on this task, it is important however to point out that in the following covariance elements will often be calculated. The evaluation of an element of such a matrix involves taking an expectation of the product of two SDEs. In general, speaking of an expectation only makes sense once the measure under which is taken is specified. In this particular case however, all the relevant measures are linked by a Radon-Nykodym derivative, and only differ (Girsanov's theorem) by a drift transformation that does not affect the product in the covariance element (formally, terms in dt dt and dt dz disappear). In order to keep notation light, in the following I will therefore omit explicit mention of the measure, implicitly assuming that any of the equivalent measures is used.

In order to achieve the task sketched above, one can begin by writing

$$\underset{(m \times 1)}{\mathbf{SR}} = \underset{(m \times m)}{\mathbf{w}} \quad \underset{(m \times 1)}{\mathbf{f}} \qquad (1)$$

where the vector **SR** contains the m co-terminal swap rates, the vector **f** represents the underlying forward rates, (i.e. the forward rates in the longest co-terminal swap), and the (m x m) matrix **w** contains the weights

w_{11}	w_{12}	w_{13}	...	w_{1m}
0	w_{22}	w_{23}	...	w_{2m}
0	0	w_{33}	...	w_{3m}
...	...	....	...	...

0 0 0 0 1 (2)

More precisely, the element w_{ij} represents the weight of the j-th forward rate in the average (Equation 1) that produces the i-th co-terminal swap rate. Notice that the m-th co-terminal swap rate coincides with the m-th forward rate: therefore $w_{mm} = 1$.

If a joint log-normal assumptions is made for the forward rates and the swap rates one can write

$$df_r/f_r = \mu_{fr}\, dt + \sigma^f_r\, dz_{fr} \quad (3)$$

$$dSR_i/SR_i = \mu_{SRi}\, dt + \sigma^{SR}_i\, dz_{SRi} \quad (4)$$

In the expression above σ^f_r and σ^{SR}_i , denote the instantaneous volatility for forward rate r and swap rate i, respectively. As is well known (see, e.g. Rebonato (1999b), Jamshidian (1997), and Rebonato and Jaeckel (2000b)), the assumption of joint log-normality for a given swap rate and the underlying forward rates and of joint deterministic volatilities are incorrect. Furthermore, and also incorrectly, we are assuming that the forward-rate and swap-rate volatilities can be assumed to be simultaneously deterministic (a pure function of time and maturity). All the results presented in this paper hinge on the validity of these approximations, whose excellent quality has been discussed in detail in Rebonato and Jaeckel (2000). Note also that, at this stage, we have not chosen the Brownian increments to be un-correlated, i.e., with hopefully obvious notation,

$$E\,[\, dz_i^f\, dz_j^f] = \rho_{ij}^f\, dt$$

and, similalry,

$$E\,[\, dz_i^{SR}\, dz_j^{SR}] = \rho_{ij}^{SR}\, dt$$

Given the definitions above, let us evaluate the instantaneous covariance between two generic swap rates, say, the i-th and the j-th, belonging to same co-terminal set. From the definitions above we can write[3]

$E[\ dSR_i/SR_i\ dSR_j/SR_j\ ]\ =$

$E[\sum_{r=1,n(i)} w_{ir}\ df_r/\sum_{r=1,n(i)} w_{ir}\ f_r \quad \sum_{s=1,n(j)} w_{js}\ df_s/\sum_{s=1,n(j)} w_{js}\ f_s] =$

$E[\sum_{r=1,n(i)} w_{ir}\ f_r \sigma^f_r\ dz^f_r\ /\sum_{r=1,n(i)} w_{ir}\ f_r \sum_{s=1,n(j)} w_{js}\ f_s \sigma^f_s\ dz^f_s\ /\sum_{s=1,n(j)} w_{js}\ f_s] =$

$$E[\sum_{r=1,n(i)}\sum_{s=1,n(j)} w_{ir}\ f_r \sigma^f_r\ /[\sum_{r=1,n(i)} w_{ir} f_r]\ \ w_{js}\ f_s \sigma^f_s\ /[\sum_{s=1,n(j)} w_{js} f_s]\ \ \rho_{rs}] \qquad (5)$$

where n(i) is the number of forward rates in the i-th swap. If one then defines

$\zeta_r^{\ i} = w_{ir}\ f_r/\sum_{r=1,n(i)} w_{ir}\ f_r\ .$

one can write more concisely

$E[\ dSR_i/SR_i\ dSR_j/SR_j\ ]\ =$

$$=[\sum_{r=1,n(i)} \sum_{s=1,n(j)} \zeta_r^{\ i}\ \zeta_s^{\ j}\ \sigma^f_r\ \sigma^f_s\ \rho_{rs}] \qquad (6)$$

or, in matrix form,

$$\text{covar}\ [dSR/SR] = \mathbf{Z}\ \mathbf{S}^{\mathbf{f}}\ \boldsymbol{\beta}\ \boldsymbol{\beta}^{\mathbf{T}}\ \mathbf{S}^{\mathbf{fT}}\ \mathbf{Z}^{\mathbf{T}} \qquad (7)$$

where the superscript 'T' indicates the transpose of a matrix, **Z** is the (n x n) matrix containing the weights {ζ}, n is the number of co-terminal swaps, $\boldsymbol{S}^f$ the (n x n) diagonal matrix containing on the main diagonal the elements the instantaneous volatility of the different forward rates, $\sigma^f_{\ r}$, $1 \leq r \leq n$, i.e.

[3] If one wanted to include the improvements to the above-mentioned approximations discussed in Rebonato and Jaeckel (2000) , one should simply replace the weight w_i in the numerator of Equation (5) with the appropriate quantity derived in detail in the above-mentioned paper.

$$
S^f = \begin{pmatrix} \sigma^f_1 & 0 & 0 & \dots & 0 \\ 0 & \sigma^f_2 & 0 & \dots & 0 \\ 0 & 0 & \sigma^f_3 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & \sigma^f_n \end{pmatrix}
$$

and

$$\boldsymbol{\beta\beta}^T = \boldsymbol{\rho}. \qquad (8)$$

Notice that the matrix $\boldsymbol{\beta}$ is uniquely defined if it is required to be symmetric and positive definite. We shall obtain explicitly the components of $\boldsymbol{\beta}$ in the following.

In order to establish the correspondence between the swap-rate/swap-rate covariance matrix and the forward-rate/forward-rate covariance matrix, it is useful to rewrite the equations that translate the log-normal character of the forward and swap rates in terms of orthogonal Brownian increments (denoted by dZ in the rest of the paper). For maximum generality we shall retain as many factors as forward rates. Also, we neglect in the following the drift terms stemming from the no-arbitrage conditions, since the terms in dt will prove irrelevant for the computation of the covariances. With these choices one can write

$$df_r/f_r = \sigma^f_{r1}\, dZ_1 + \sigma^f_{r2}\, dZ_2 + \sigma^f_{r3}\, dZ_3 + \dots + \sigma^f_{rn}\, dZ_n \qquad 1 \le r \le n \quad (9)$$
$$= \sum_{m=1,n(i)} \sigma^f_{rm}\, dZ_m \qquad 1 \le m \le n$$

With the notation chosen, the different number of subscripts indicates whether the symbol σ is to be understood in the sense of Equation (1) or (9). In matrix form the above can be written

$$\mathbf{[df/f]} = \sigma^f \boldsymbol{dZ}$$

with

$$\sigma^f = \begin{matrix} \sigma^f_{11} & \sigma^f_{12} & \sigma^f_{13} & \ldots & \sigma^f_{1n} \\ \sigma^f_{21} & \sigma^f_{22} & \sigma^f_{23} & \ldots & \sigma^f_{2n} \\ \ldots & & & & \\ \sigma^f_{n1} & \sigma^f_{n2} & \sigma^f_{n3} & \ldots & \sigma^f_{nn} \end{matrix}$$

By dividing and multiplying each term in Equation (9) by the quantity $\sum_{m=1,n(i)} \sigma^f{}_{rm}{}^2$ one can write

$$df_r/f_r = [\sum_{m=1,n(i)} \sigma^f{}_{rm}{}^2] \sum_{m=1,n(i)} \sigma^f{}_{rm} / [\sum_{m=1,n(i)} \sigma^f{}_{rm}{}^2] \, dZ_m$$

However, given the orthogonality of the increments dZ_s,

$$[\sum_{m=1,n(i)} \sigma^f{}_{rm}{}^2] = \sigma^f{}_r \qquad (10)$$

Therefore, if one defines

$$\sigma^f{}_{rm} /[\sum_{m=1,n(i)} \sigma^f{}_{rm}{}^2] = b^f{}_{rm} \qquad (11)$$

one can re-write Equation (12)

$$df_r/f_r = \sigma^f{}_r \sum_{m=1,n(i)} b^f{}_{rm} \, dZ_m$$

If one now defines by $\mathbf{B}^f$ the (n x n) matrix containing the elements $\{b^f{}_{rm}\}$ from Equation (13) and makes use of the definition of the $\mathbf{S}^f$ matrix presented above, one can write

$$\underset{[n \times 1]}{[\mathbf{df/f}]} = \underset{[n \times n]}{\mathbf{S}^f} \; \underset{[n \times n]}{\mathbf{B}^f} \; \underset{[n \times 1]}{\mathbf{dZ}} \qquad (12)$$

Equation (12) expresses in a convenient matrix form the stochastic part of the dynamics of the forward rates. Exactly the same reasoning could have been applied to swap rates rather than forward rates. In the notation above the superscript ‘f’ would simply have to be replaced by ‘SR’. Notice that, since one set of variables (say, the swap rates) can be exactly expressed as a function of the other set (say, the forward rates) exactly the same orthogonal Brownian increments shock both sets of quantities. The only quantity that changes in moving from forward to swap rates are the loadings b^{f}_{rm}, b^{SR}_{jm},. Therefore expression (11) above can be re-written by inspection as

$$[\mathbf{dSR/SR}] = \mathbf{S}^{SR}\,\mathbf{B}^{SR}\,\mathbf{dZ}$$

In particular, the swap-rate/swap-rate covariance matrix can be written as

$$\mathrm{covar}[dSR/SR] = \mathbf{S}^{SR}\,\mathbf{B}^{SR}\,\mathbf{B}^{SR\mathbf{T}}\,\mathbf{S}^{SR\mathbf{T}}$$

where use has been made again of the fact that $\mathbf{dZ}\,\mathbf{dZ}^{T} = \mathbf{I}$.

Equating this expression with the equation previously obtained for the swap-rate/swap-rate covariance matrix in terms of forward rate volatilities and correlation, (Equation (10’)) one can write

$$\mathbf{S}^{SR}\,\mathbf{B}^{SR}\,\mathbf{B}^{SR\mathbf{T}}\,\mathbf{S}^{SR\mathbf{T}} = \mathbf{Z}\,\mathbf{S}^{\mathbf{f}}\,\boldsymbol{\beta}\,\boldsymbol{\beta}^{\mathbf{T}}\,\mathbf{S}^{\mathbf{fT}}\,\mathbf{Z}^{\mathbf{T}}.$$

However, form the definitions above, and by direct calculation one can easily see that $\mathbf{B}\,\mathbf{B}^{T} = \rho$, and therefore $\mathbf{B} = \boldsymbol{\beta}$. It therefore follows that

$$\mathrm{covar}[dSR/SR] = \boldsymbol{\sigma}^{SR}\,(\sigma^{SR})^{\mathbf{T}} \qquad (13)$$

$$\mathrm{covar}[df/f] = \boldsymbol{\sigma}^{f}\,(\sigma^{f})^{\mathbf{T}} \qquad (14)$$

$$\boldsymbol{\sigma}^{SR}\,(\sigma^{SR})^{\mathbf{T}} = \mathbf{Z}\,\boldsymbol{\sigma}^{f}\,(\sigma^{f})^{\mathbf{T}}\,\mathbf{Z}^{\mathbf{T}}. \qquad (15)$$

$$\boldsymbol{\sigma}^{f}\,(\sigma^{f})^{\mathbf{T}} = \mathbf{Z}^{-1}\,\boldsymbol{\sigma}^{SR}\,(\sigma^{SR})^{\mathbf{T}}\,(\mathbf{Z}^{\mathbf{T}})^{-1} \qquad (16)$$

These results give the elements of the (instantaneous) covariance matrix between co-terminal swap rates in terms of the weights, $\boldsymbol{\zeta}$, and of the elements of the (instantaneous) covariance matrix between forward rates. These simple relationships allow the direct translation form the stochastic dynamics of one set of variables (say, forward rates) to the complementary set (swap rates).

Needless to say, the crucial assumption in obtaining these results is that the quantities $\mathbf{Z}$ can be treated as constants while performing the expectations. Given the very good quality of these approximation, Equations (13) to (16) then provide tools for evolving forward rates in such a way that an arbitrary set of market values for the corresponding co-terminal swaptions is (almost) exactly recovered. The details of the procedure are given in the following section, that deals with the problem of the valuation of Bermudan swaptions. The important point to stress, however, is that it will become apparent that the almost exact recovery of the European swaptions prices by evolving forward rather than swap rates can be achieved in an infinity of ways. I will supply criteria to choose between these possible solutions on the basis of financial judgement. Producing the forward rate dynamics capable of reflecting as accurately as possible these financial desiderata will be shown to be equivalent to specifying the forward-rate instantaneous volatility functions. If these are parametrized, perhaps as described in Rebonato (1999c), the ability to perform a numerical search in this parameter space under the constraint of the correct recovery of the European swaption prices will hinge crucially on the availability of quick and efficient ‘translation rules’ such as the ones in Equations (13) to (16). How this can be accomplished is detailed in the next section.

3 – An application: Pricing Bermudan swaptions

In order to see how the results presented in the previous section can be profitably used in practice, one can turn to the evaluation of Bermudan swaptions in a BGM framework. As discussed above the first and most important desideratum is, of course, the exact recovery of the prices of the co-terminal Euroepan swaptions. However, many specifications of the swap-rate instantaneous volatility functions can produce this result, and yet give rise to significantly different prices for the underlying Bermudan swaption. See, e.g., the discussion in Sidenius (2000). It is therefore essential to ensure that, given the correct pricing of the European swaptions, the resulting forward rate dynamics should be as realistic and convincing as possible. Furthermore, the prices of the caplets are of course important, but the implied forward-rate/forward-rate and swap-rate/swap-rate correlations also play a significant role. This is exactly where the links established above become extremely useful. The task can be accomplished as follows.

To begin with, from the market values of European swaptions one can choose the instantaneous volatilities of the swap rates so that the integrals of their squares fit the market. Let us assume that the trader has chosen a set of instantaneous volatility functions $\sigma^{SR}{}_i(t)$ such that their root mean square out to the expiry of the relative European swaption coincides with the corresponding implied volatility. In other words, we shall assume in the following that the trader has already made his choice about the instantaneous volatility of the each co-terminal swap rates $\sigma^{SR}{}_i(t)$. After choosing the number s of orthogonal factors, one can write

$$dSR_i = \mu_{SRi}\, dt + \sum_{r=1,s} \sigma^{SR}{}_{ir}\, dz_r \qquad (17)$$

One can then make use again of the trigonometric decomposition described in Rebonato (1999a) and Rebonato and Jaeckel (2000b): for two factors this is equivalent to writing

$$dSR_i = \mu_{SRi}\, dt + \sigma_{SRi}\, [\cos(\theta_i)\, dz_1 + \sin(\theta_i)\, dz_2] \qquad (18)$$

Clearly this specification of the loadings onto the different Brownian increments of the instantaneous volatility of the swap rate i will always yield the correct instantaneous volatility because, for any x, $\sin^2(x) + \cos^2(x) = 1$. More generally, in higher dimensions, this result can be generalized as

$$dSR_i = \mu_{SRi}\, dt + \sum_{r=1,s} \sigma^{SR}{}_{ir}\, [b_{ik}\, dz_k]$$

with

$\sum_{r=1,s} b_{ik}^2=1$ (19)

and by choosing the coefficicients {b} such that

$$b_{ik}(t) = \cos(\theta_{ik}(t)) \Pi_{j=1,k-1} \sin(\theta_{ij}(t)), \quad k=1,s-1$$

$$b_{ik}(t) = \Pi_{j=1,k-1}\sin(\theta_{ij}(t)), \quad k=s \quad (20)$$

For s factors the coefficients {**b**} assume the obvious interpretation of the angular co-ordinates of an s-dimensional unit-radius sphere. The important point to notice at this stage is that, for *any* choice of these angles, all the swap-rate instantaneous volatilities are exactly recovered and, therefore, the co-terminal European swaptions are always perfectly priced. At the same time, each choice of angles will uniquely determine the covariance matrix of the forward rates (and therefore the caplet prices and the forward-rate/forward-rate correlation matrix). The advantage of making use of these trigonometric relationships is that the resulting optimization is totally unconstrained. See again and Rebonato (1999c) and Rebonato and Jaeckel (2000b) for a more thorough discussion of this point.

It is worthwhile noting in passing that, for any choice of the angles {θ}, the quantities b_{ik} can assume values between –1 and 1. Furthermore, one can easily prove that the matrix B of elements b_{ik} is such that $\mathbf{B}\,\mathbf{B}^\mathbf{T}$ is always positive definite, for any choice of {θ}. Therefore the matrix defined by Equation (20) is a possible candidate for the matrix $\boldsymbol{\beta}$ in Equation (8). This clearly shows that choosing the angles in Equation (20) is simply a numerically efficient way of specifying a possible correlation matrix.

Going back to the calculation at hand, for s factors, (s-1) vectors, each made up of n angles, will 'contain' the quantities $\{b_{ik}\}$.One can begin by choosing the angle vectors at random. This will produce a matrix $\boldsymbol{\sigma}^{SR}$. By making use of Equation (13), from this one can calculate the covariance matrix element between swap rates simply as

$$\text{covar}[dSR/SR] = \boldsymbol{\sigma}^{SR} (\sigma^{SR})^\mathbf{T}.$$

Given the relationships established in the previous sections one can immediately evaluate the covariance matrix between the forward rates as a function of these initial random angles. More precisely, by Equations (16) and (18) one can write

$$\text{covar}\ [\mathbf{df/f}] = \boldsymbol{\sigma}^{f} (\sigma^{f})^{\mathbf{T}} = \mathbf{Z}^{-1}\ \boldsymbol{\sigma}^{SR}\ \sigma^{SR\mathbf{T}}\ (\mathbf{Z}^{\mathbf{T}})^{-1} \qquad (21)$$

By making use of this relationship, and by carrying out just a few matrix multiplications, one can immediately obtain the forward-rate/forward-rate covariance matrix as a function of the initial (possibly random) choice for the angles {θ}. In particular, the elements on the main diagonal are directly linked to the caplet volatilities, and allow immediate comparison with market observables. More generally, one can extract both the instantaneous volatilities of and the correlations among the forward rates, and comment on their plausibility. If this were found wanting, one could iteratively vary the angles (which had originally been chosen at random) by requiring that the discrepancies between a desired forward/forward (instantaneous) covariance matrix and $\mathbf{Z}^{-1}\ \boldsymbol{\sigma}^{SR}\ \sigma^{SRT}\ (\mathbf{Z}^{\mathbf{T}})^{-1}$ should be as small as possible. Therefore, all the underlying caplets or the forward-rate/forward-rate covariance matrix are obtained as best as possible under the constraint that all the co-terminal European swaptions should at the same time be priced correctly.

In order to illustrate the procedure more precisely I analyze in the xample below below the stylized case of a forward-rate implementation with constant (time-independent) instantaneous volatilities. More realistic numerical experiments that illustrate the quality of the approximations and the practical applicability of the approach are presented in the following and last section of the paper.

Section 4 – An example: Obtaining Swap-Rate volatilities while fitting to the FRA-covariance matrix.

In order to illustrate the procedure the case is considered in this section of a 5-forward-rate, 5-co-terminal-European-swaption situation. The swap and forward rate volatilities are assumed to be exogenously available from the market. They are displayed in the columns VolFRA and VolSwap. The five semi-annual forward rates and the spot rate are shown in the column labelled ‘FwdRates’. The column ‘SwapRates’ contains the various co-terminal swap rates; the first entry is the (spot) swap rate from today to the final maturity. The other columns have an obvious meaning. For simplicity the instantaneous volatilities of the forward rates are assumed to be constant (i.e. maturity-dependent but time-independent)

The purpose of the exercise is to produce the an almost exact recovery of the swap-rate volatilities while, at the same time, producing the ‘best’ possible fit to selected portions of the forward-rate/forward-rate covariance matrix. I intend to highlight how different choices for the target (e.g. caplets *versus* the whole matrix) can give rise to radically different solutions.

Time	FwdRates	Discount	Swaprates	VolFRA	VolSwap
0	6.00%	1	6.456%		
0.5	6.25%	0.970873786	6.557%	**20.00%**	**18.31%**
1	6.50%	0.941453369	6.640%	**22.00%**	**18.68%**
1.5	6.65%	0.911819243	6.689%	**20.00%**	**18.21%**
2	6.70%	0.882476887	6.710%	**19.00%**	**18.19%**
2.5	6.72%	0.853872169	6.720%	**18.50%**	**18.50%**
3		0.826114715			

Tab I: The case study. See the text for a description of the labelling of the columns.

Following the procedure described in Section 3 the angles {θ} were first of all varied in such a way as to produce the best possible fit to the trace of $\boldsymbol{\sigma}^{\mathbf{f}}\,\boldsymbol{\sigma}^{\mathbf{f}T}$ by making use of the trigonometric relationships (20) and of Equation (21). The simple functional form for the correlation function (i.e. $\rho=\exp[-\beta\,|t_i-t_j|]$, with $\beta=0.1$) was chosen for the exercise. Given the assumption of constant instantaneous volatility, imposing the best fit to the trace of the matrix $\boldsymbol{\sigma}^{\mathbf{f}}\,\boldsymbol{\sigma}^{\mathbf{f}T}$ is equivalent to ensuring the best possible fit to the caplet prices. Indeed, after varying over the angles, the resulting fit to the caplet prices turned out to be virtually perfect, as shown in table II below.

Target	Model
20.000%	20.000%
22.000%	22.000%
20.000%	20.000%
19.000%	19.000%
18.500%	18.500%

Tab.II: Target and model caplet prices in the case of a fit to the trace of $\sigma^f \sigma^{fT}$

By ‘forcing’ the fit to a specific portion of the covariance matrix (i.e. to its main diagonal), however, both the resulting forward-rate/forward-rate and swap-rate/swap-rate correlation matrices turned out to be highly implausible. A cross-section of these matrices is shown in Figs. 1 and 2 below.

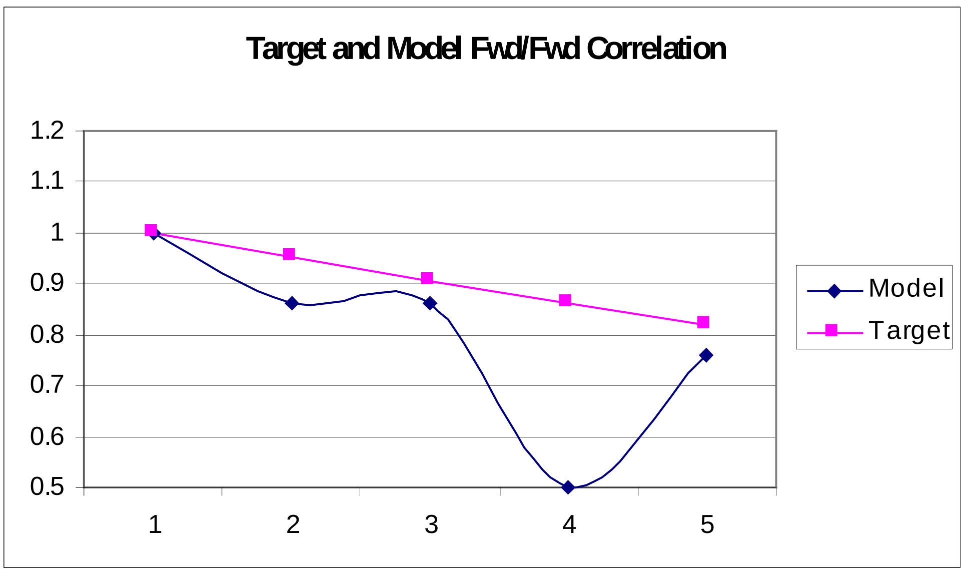

Fig. 1: The forward-rate/forward-rate correlation produced by the best fit to the main diagonal of the forward-rate/forward-rate covariance matrix. The line labelled ‘target’ corresponds to the correlation function $\rho=\exp[-\beta\,|t_i-t_j|]$, with $\beta=0.1$.

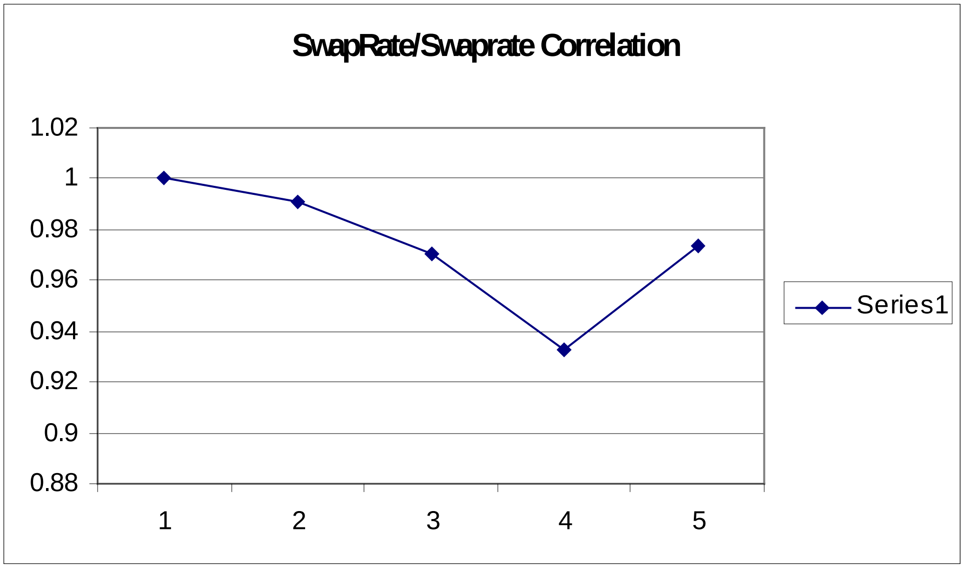

Fig. 2: The swap-rate/swap-rate correlation produced by the best fit to the main diagonal of the forward-rate/forward-rate covariance matrix.

Clearly, the resulting correlation functions are implausible to say the least, and are the by-product of attempting 'too good' a fit of some sub-set of the target surface. This should come as no surprise given the related results reported in Rebonato (1999a).

These findings can be contrasted with the results that can be obtained if, instead, one attempts an overall fit to the whole covariance matrix. The prices of the individual caplets are now no longer exactly recovered. (See Tab. III and Fig. 3). The discrepancy, however, between model and target prices turns out to be very small. More importantly, the resulting forward-rate/forward-rate and swap-rate/swap-rate correlation matrices now display a much more plausible and desirable behaviour. This is shown in the tables (III and IV) and figures (4 and 5) below.

Target	Model
20.00%	19.64%
22.00%	22.05%
20.00%	19.78%
19.00%	18.86%
18.50%	18.50%

Tab.III: Model and Target caplet Volatilities in the case of a fit to the whole forward-rate/forward-rate covariance-matrix.

Swaprate/Swaprate Correlation Matrix				
1	**0.99434**	**0.97846**	**0.95044**	**0.90855**
0.99434	**1**	**0.99222**	**0.9698**	**0.93075**
0.97846	**0.99222**	**1**	**0.98882**	**0.95596**
0.95044	**0.9698**	**0.98882**	**1**	**0.98008**
0.90855	**0.93075**	**0.95596**	**0.98008**	**1**

Tab IV: The swap-rate/swap-rate correlation matrix obtained with a fit to the whole forward-rate/forward-rate covariance-matrix

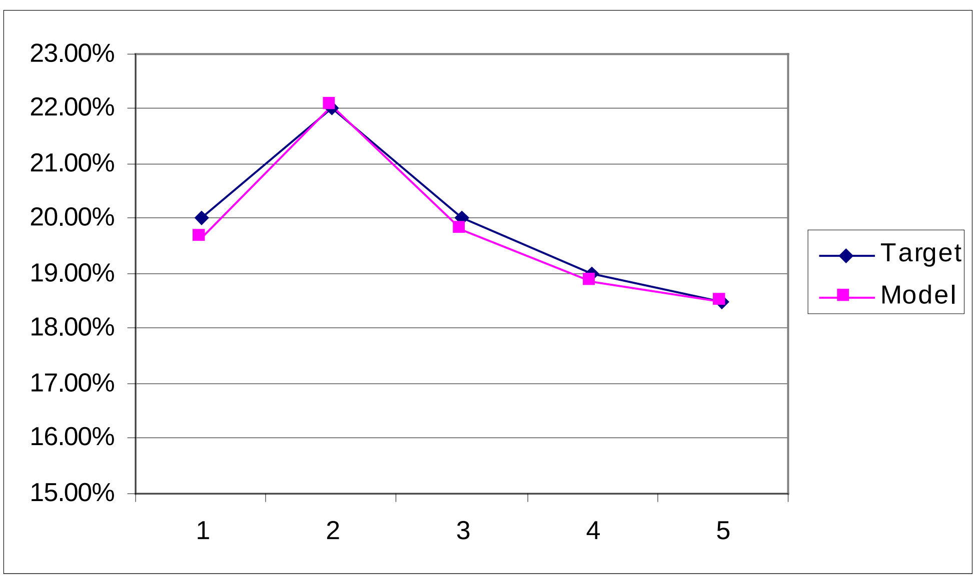

Fig. 3: Target and model caplet prices in the case of a fit to the whole matrix $\sigma^f \sigma^{fT}$

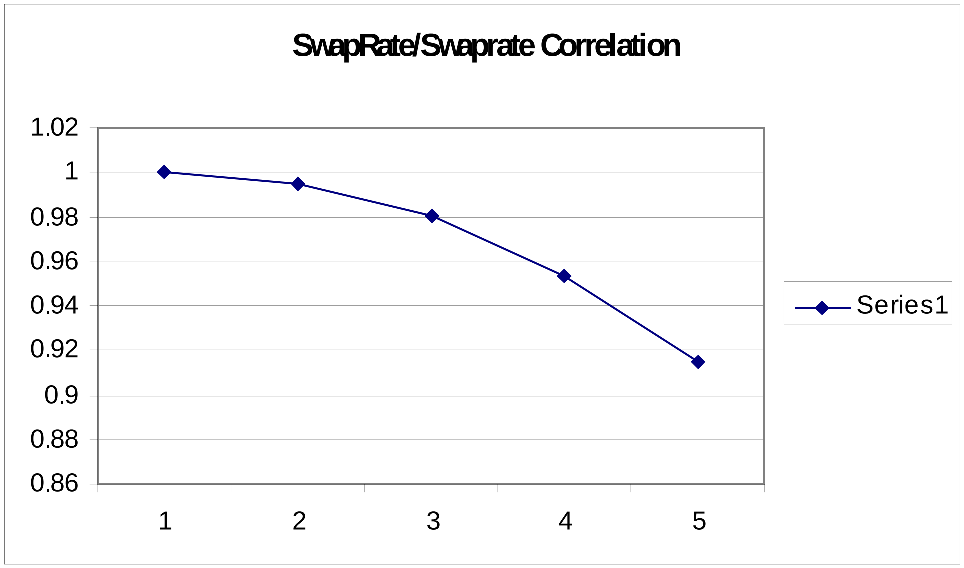

Fig. 4: The swap-rate/swap-rate correlation produced by the best fit to the whole forward-rate/forward-rate covariance matrix.

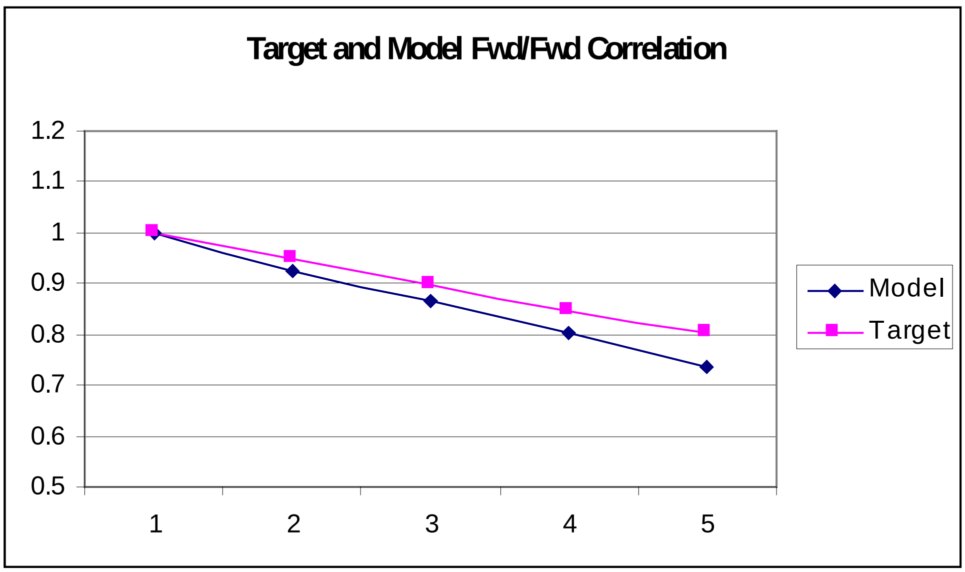

Fig. 5: The forward-rate/forward-rate correlation produced by the best fit to the whole forward-rate/forward-rate covariance matrix.

The main message of the example treated in this section is therefore that an efficient fit can indeed be obtained to selected portions of an exogenous forward-rate/forward-rate covariance matrix while recovering the prices of the European co-terminal swaptions; it was also shown, however, that focussing on specific areas of the target covariance matrix can be dangerous, the more so if the caplet and swaption markets are not highly congruous. A modest price in terms of caplet mispricing can be more than offset by a much more plausible correlation structure between the various variables.

What this example has not shown is the quality of the recovery of the European swaption prices in realistic market cases. This task is undertaken in the next section of this paper.

5 – Quality of the numerical approximation: the recovery of European swaption prices.[4]

This section illustrates how well the European swaption prices can indeed be recovered while fitting to the forward-rate/forward-rate covariance matrix in a realistic market case. The functional form for the instantaneous volatilities of and correlation among the forward rates was assumed to be (see Rebonato (1999c) for a discussion)

$$\sigma_{inst}^{i}(t) = k_i \{[a + b (T_i - t)] \exp(-c (T_i - t) + d\}$$

$$\rho_{ij} = \exp(-\beta |T_i - T_j|)$$

where T_i is the expiry of forward rate i, and $\sigma_{inst}^{i}(t)$ its instantaneous volatility at time t. A market yield curve, swaption matrix and term structure of volatilities were then chosen (GBP June 2000), and the case of a 15-year non-call-10 Bermudan swaption[5] was then considered. The relevant market information is displayed in the Tables I and II below.

[4] It is a pleasure to acknowledge the help of Dr. Hunter and Dr. Jaeckel in producing the computational tools used to obtain the results presented in this section.

[5] The expression 'n non-call m Bermudan swaption' indicates a Bermudan swaption that cannot be exercised for the first m years, and whose final maturity is n years from today.

Underlying Forward Rates				Co-Terminal Swaps			
Reset Time(yrs)	Pay Time(yrs)	Accrual Factor	Forward Rate	Market Swpn Vols	Swap Rate	Swaption Price	Swaption Vega
10.0	10.50	0.50	6.21%	15.5%	5.94%	3,551,759	139,983
10.5	11.00	0.50	6.21%	16.0%	5.91%	3,202,308	127,467
11.0	11.50	0.50	6.15%	16.4%	5.86%	2,842,528	114,333
11.5	12.00	0.50	6.15%	17.1%	5.82%	2,507,729	100,666
12.0	12.50	0.50	5.80%	17.8%	5.76%	2,151,316	86,566
12.5	13.00	0.50	5.80%	18.7%	5.75%	1,856,393	72,211
13.0	13.50	0.50	5.79%	19.7%	5.73%	1,533,658	57,683
13.5	14.00	0.50	5.79%	20.8%	5.71%	1,187,145	43,086
14.0	14.50	0.50	5.67%	21.6%	5.67%	805,100	28,554
14.5	15.00	0.50	5.67%	22.1%	5.67%	408,532	14,244
15.0							

Tab. I: The details of the problem analysed in the text. The labelling of the columns is self-explanatory

	0.5	1.0	2.0	3.0	4.0	5.0
10.0	20.1830%	20.8121%	19.6060%	18.8559%	18.2280%	*18.0275%*
11.0	20.7028%	21.3204%	19.8983%	19.0612%	*18.4186%*	18.2188%
12.0	20.5271%	21.1702%	20.1548%	*19.2541%*	18.7024%	18.3425%
13.0	20.9900%	21.6330%	*20.7289%*	19.7999%	19.0515%	18.6879%
14.0	21.5220%	*22.1505%*	21.6154%	20.2533%	19.5773%	18.9879%
15.0	*22.6495%*	23.3050%	22.3177%	20.9617%	20.0347%	19.4349%

Tab II: The portion of the swaption matrix relevant for the problem at hand: the values in the first column indicate the expiry times of the European swaptions, and the numbers in the first row the maturity of the swap entered into. Entries underlined and in italics indicate the volatilities of the co-terminal swaptions relevant for the chosen Bermudan.

The next table then shows the values of the optimal parameters {a,b,c,d} for the forward-rate instantaneous volatility function (Equation (7)) obtained by imposing the best possible fit to the overall forward-rate/forward-rate covariance matrix. In the exercise the value of the decay constant β in the correlation function was kept fixed at 0.15, and the coefficients a, b c, and d were chosen to be –0.0228, 0.0466, .304 and 0.1818, respectively.

By choosing the same coefficiewnts for all the forward rates the caplet prices are not exactly recovered. Also displayed in the next table are therefore the scaling factors {**k**} that would be necessary in order to ensure exact pricing of the caplets. It is important to notice that all the scaling factors (column 'VolScale') are remarkably constant, despite the fact that this criterion was did not enter the optimization. This indicates that the resulting evolution of the term structure of volatilities is approximately time homogeneous. This latter feature is both important and reassuring, since it had not been imposed in the fitting procedure.

Reset Date	Pay Date	Accrual Factor	Forward Rate	Market Caplet Vol	Model Caplet Vol	Vol Scale
10.00	10.25	0.25	6.16%	22.7%	21.0%	0.926
10.25	10.50	0.25	6.16%	22.7%	21.0%	0.924
10.50	10.75	0.25	6.16%	22.7%	20.9%	0.923
10.75	11.00	0.25	6.16%	22.7%	20.9%	0.922
11.00	11.25	0.25	6.10%	22.7%	20.9%	0.921
11.25	11.50	0.25	6.10%	22.6%	20.8%	0.920
11.50	11.75	0.25	6.10%	22.6%	20.8%	0.919
11.75	12.00	0.25	6.10%	22.6%	20.8%	0.917
12.00	12.25	0.25	5.76%	22.5%	20.7%	0.923
12.25	12.50	0.25	5.76%	22.4%	20.7%	0.922
12.50	12.75	0.25	5.76%	22.4%	20.7%	0.922
12.75	13.00	0.25	5.76%	22.4%	20.6%	0.921
13.00	13.25	0.25	5.74%	22.4%	20.6%	0.921
13.25	13.50	0.25	5.74%	22.3%	20.6%	0.921
13.50	13.75	0.25	5.74%	22.3%	20.5%	0.921
13.75	14.00	0.25	5.74%	22.3%	20.5%	0.920
14.00	14.25	0.25	5.63%	22.2%	20.5%	0.923

Tab IV: The caplet implied volatilities obtained by the optimization procedure and the scaling factors {**k**} that would be needed to bring the model prices in line with the corresponding market values.

After performing the optimization over the coefficients of the forward rate instantaneous volatility function, an 'exact' Monte Carlo evaluation of the European swaption prices was carried out retaining as many factors as forward rates in the problem (i.e. 10). The price thus obtained was then compared with the corresponding Black price. Table V then presents the main results, i.e. the quality of the recovery of the Euroepan swaption prices obtained by evolving the forward rates:

European (Black)	European (BGM)	Swaption Vega	Error (%)	Error (Fraction of Vega)
3,551,759	3,553,400	139,983	-0.046%	-1.17%
3,202,308	3,204,566	127,467	-0.071%	-1.77%
2,842,528	2,841,692	114,333	0.029%	0.73%
2,507,729	2,505,399	100,666	0.093%	2.31%
2,151,316	2,149,243	86,566	0.096%	2.39%
1,856,393	1,855,077	72,211	0.071%	1.82%
1,533,658	1,533,030	57,683	0.041%	1.09%
1,187,145	1,187,068	43,086	0.006%	0.18%
805,100	804,925	28,554	0.022%	0.61%
408,532	408,373	14,244	0.039%	1.12%

Tab V: Results for the prices of the European co-terminal swaptions obtained by evolving the forward rates as described in the text. The last two columns display the error both in percentage terms and as a fraction of one vega (taken as a proxy of a typical bid/offer spread).

The agreement can be clearly seen to be excellent, both in percentage terms and as a fraction of the typical bid/offer spread (taken to be 1 vega). From Tab. IV one can see that the resulting caplet prices are indeed not perfectly priced, and that there is a systematic downward bias. If one so desired, this bias could be substantially reduced by changing the value for the decay constant β, but the resulting correlation function would assume implausibly low values. Strictly speaking, this inability to recover the caplet market with a higher degree of accuracy might theoretically be due to several factors, such as, for instance, an inadequate parametric form for the correlation or volatility functions. Despite the fact that these explanations are logically possible, one should not rue out the possibility that supply and demand conditions, coupled with relatively poor market liquidity, might act as a deterrent to the activity of arbitrageurs. In other terms, in the particular currency and date chosen for this example, the caplet and swaption markets would appear, to a certain extent, to be trading with a palpable lack of what Jamshidian (1997) refers to as ‘internal coherence’.

6 – Conclusions and extensions

I have shown that it is possible to specify the BGM-dynamics of forward rates in such a way that all the co-terminal European swaptions underlying a given Bermudan swaption can be almost exactly recovered, while, at the same time, obtaining as good a fit as possible to an exogenously specified forward-rate covariance matrix.

The technique can clearly be extended along similar lines to other products (such as CMS-based instruments), and should find applicability whenever the joint dynamics of swap and forward rates is an important determinant of their price, as discussed, for instance, in Jamshidian (1997).

The procedure is numerically efficient, allowing as it does to carry out an *unconstrained* optimization in parameter space. I have also shown that the results are of high numerical quality.

Work is in progress to explore by means of this technique the issue of the effect of model dimensionality on the price of Bermudan swaptions (see, e.g. Brace and Womersley (2000)).

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