A Two-Regime, Stochastic Volatility Extension of the LIBOR Market Model: Theory and Empirical Evidence

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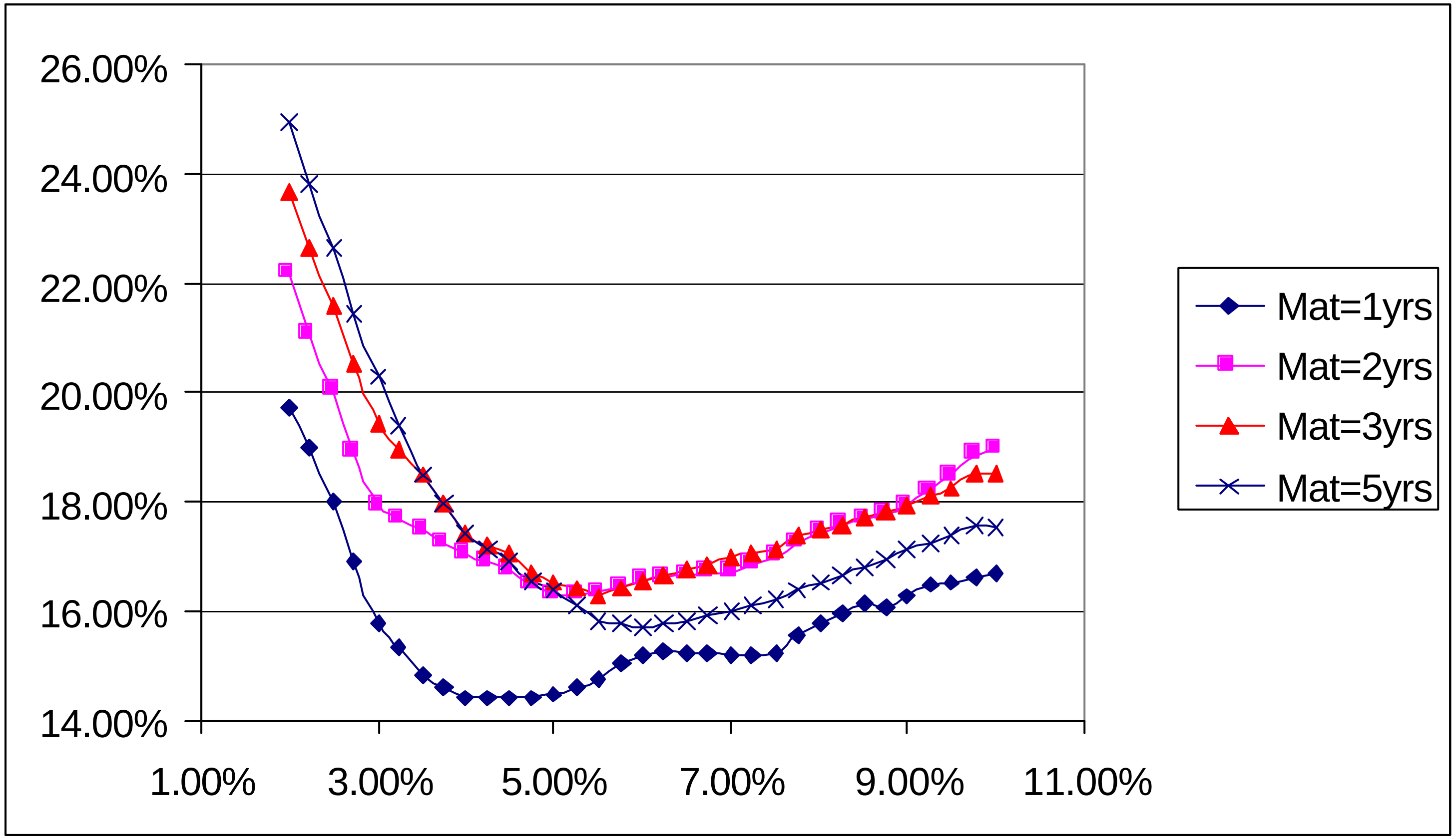
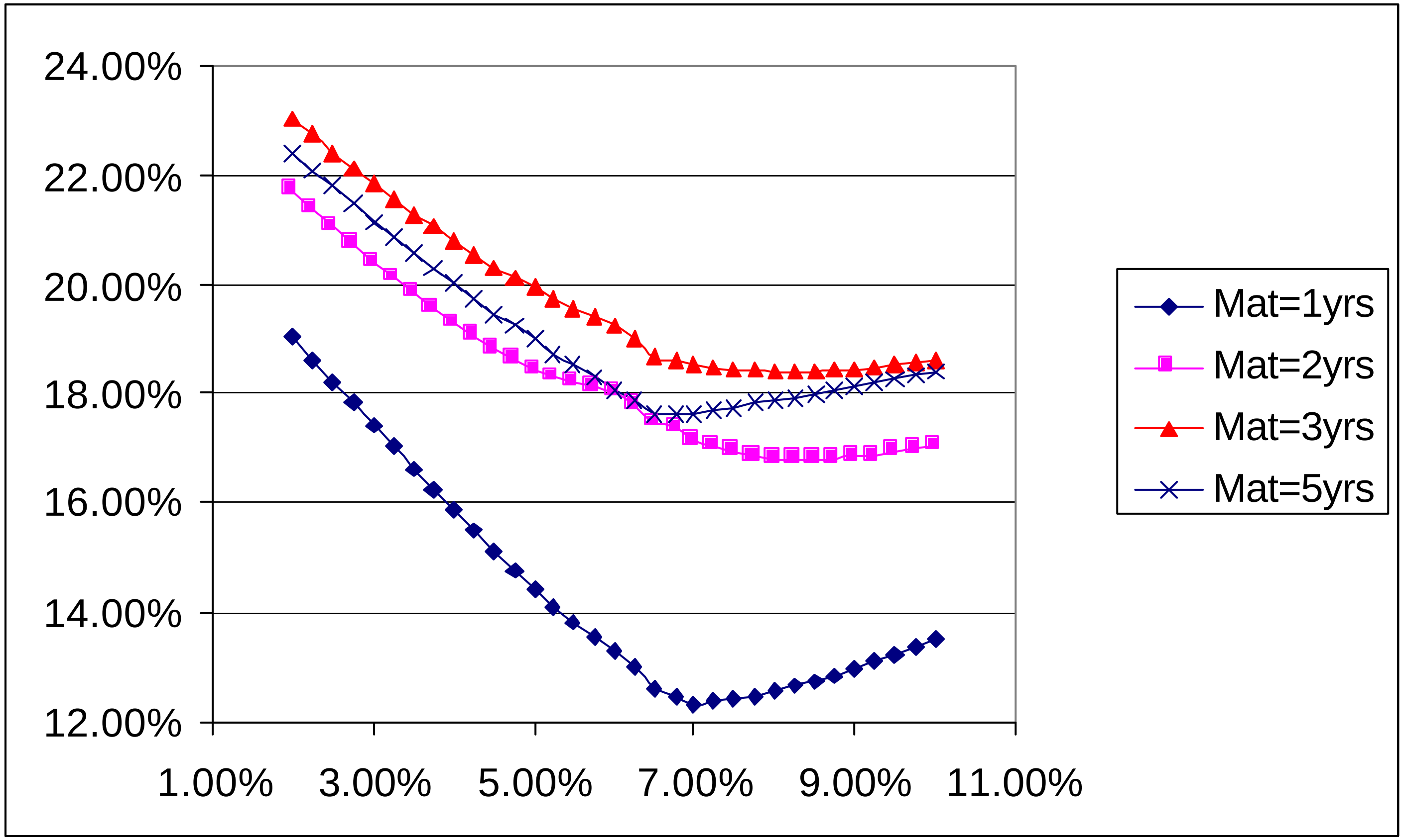
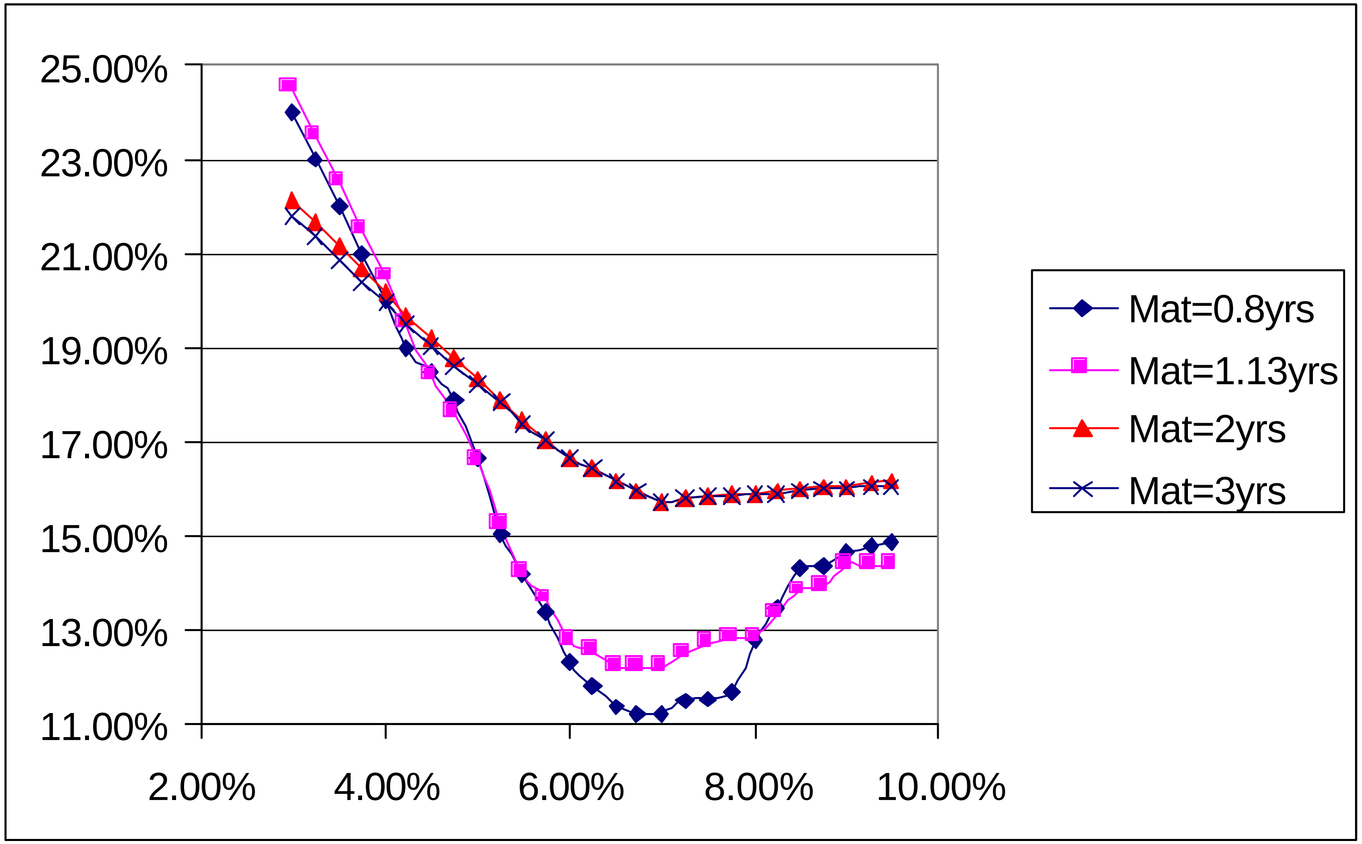
The New Smiles: US$



The New Smiles: EUR



The New Smiles: GBP



The proposed solution (Take 1)

Stochastic-Volatility Extension of the LMM:

$$\sigma_{inst}(t, T) = k(T)g(T - t)$$

$$g(T - t) = [a + b(T - t)]exp(-c(T - t)) + d$$

The proposed dynamics

$$\frac{d(f_i+\alpha)}{f_i+\alpha} = \mu_i^{\alpha}(\{f\},t)dt + \sigma_i^{\alpha}(t,T_i)dz_i$$

$$\sigma_i^{\alpha}(t,T_i) = [a_t + b_t(T-t)]exp(-c_t(T-t)) + d_t$$

$$da_t = RS_a(RL_a - a_t)dt + \sigma_a dz_a$$

$$db_t = RS_b(RL_b - b_t)dt + \sigma_b dz_b$$

$$d\ln[c_t] = RS_c(RL_c - \ln[c_t])dt + \sigma_c dz_c$$

$$d\ln[d_t] = RS_d(RL_d - \ln[d_t])dt + \sigma_d dz_d$$

$$E[dz_i dz_a] = E[dz_i dz_b] = E[dz_i dz_c] = E[dz_i dz_d] = 0$$

$$E[dz_a dz_b] = E[dz_a dz_c] = E[dz_a dz_d] = 0$$

$$E[dz_b dz_c] = E[dz_b dz_d] = 0$$

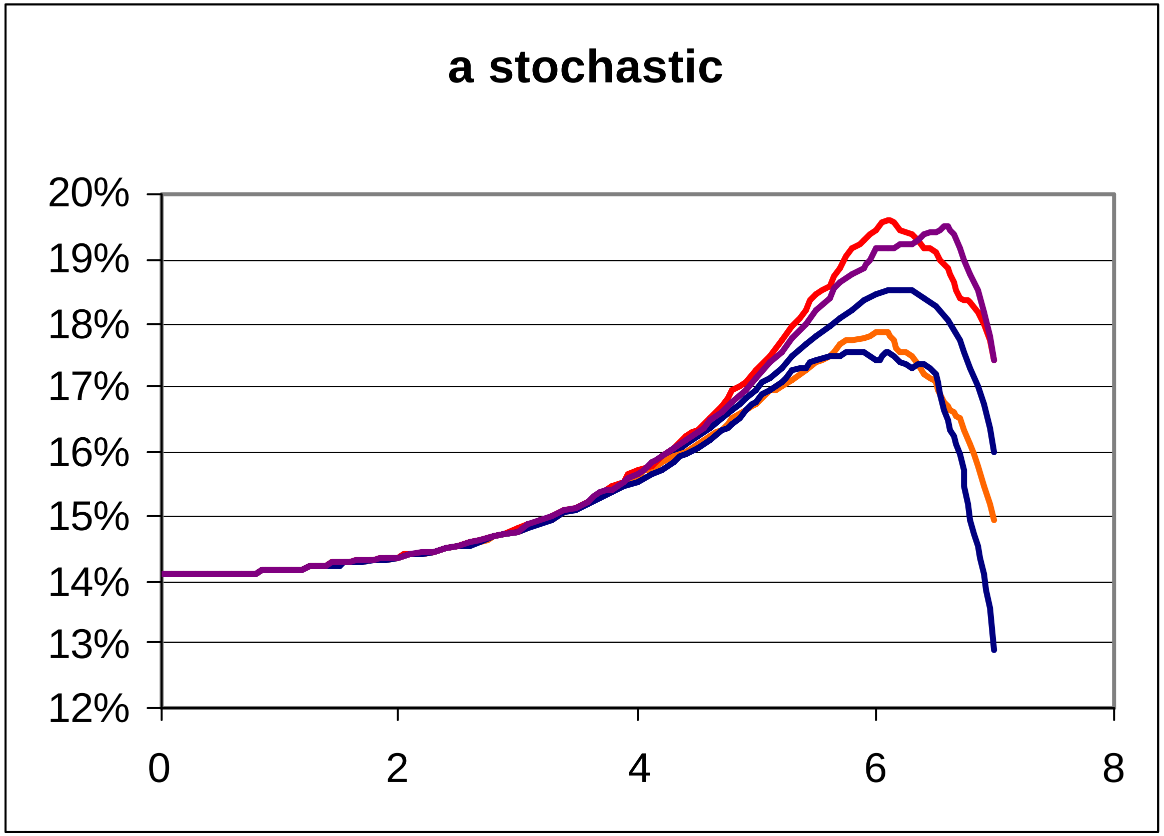
$$E[dz_c dz_d] = 0$$

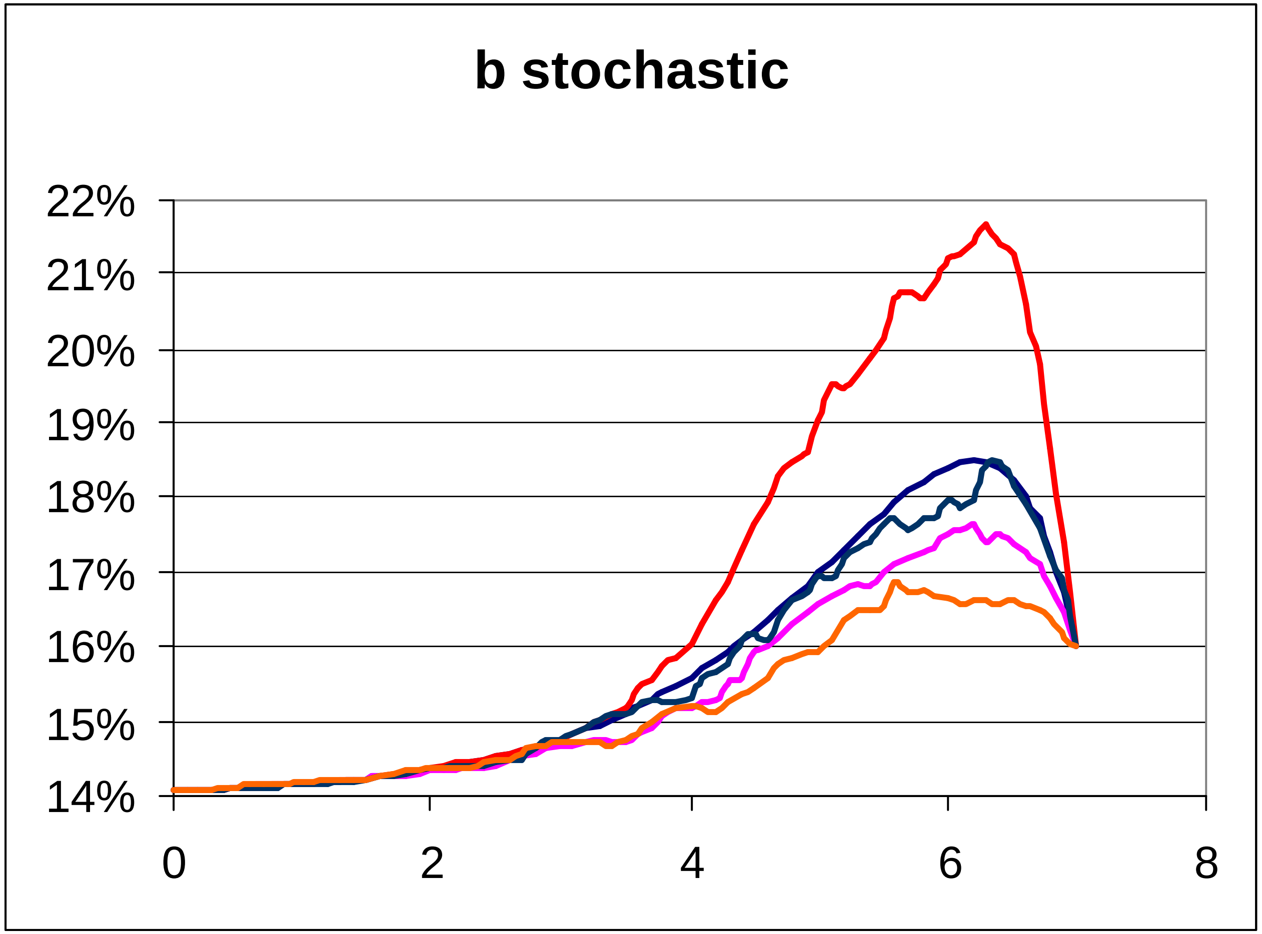
The crucial point

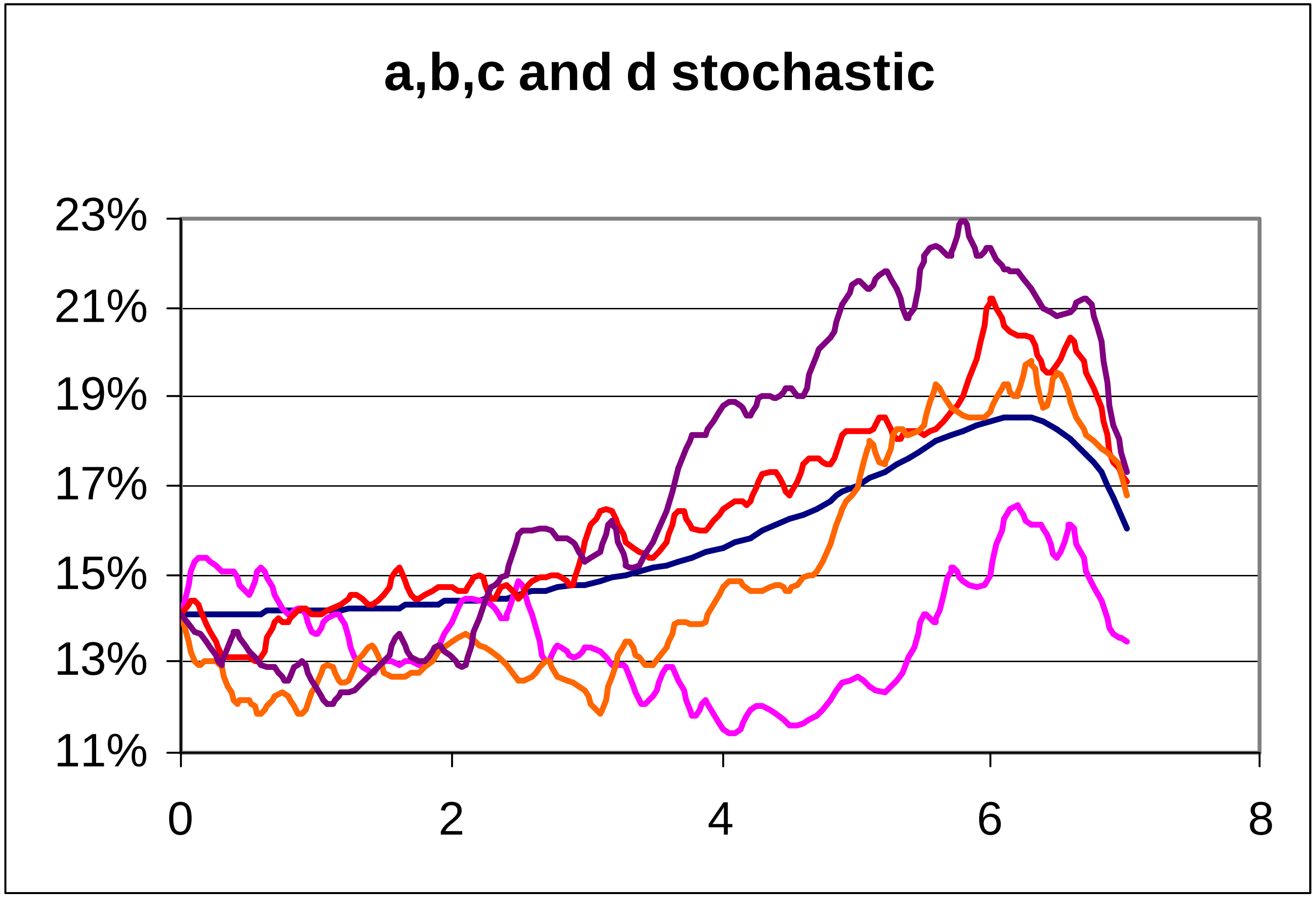
The stochastic drivers for the volatility and independent of the drivers of the forward rates. This allows to

- calibrate to caplets
- calibrate to European swaptions
- calibrate to co-terminal swaptions
- price derivatives easily and efficiently

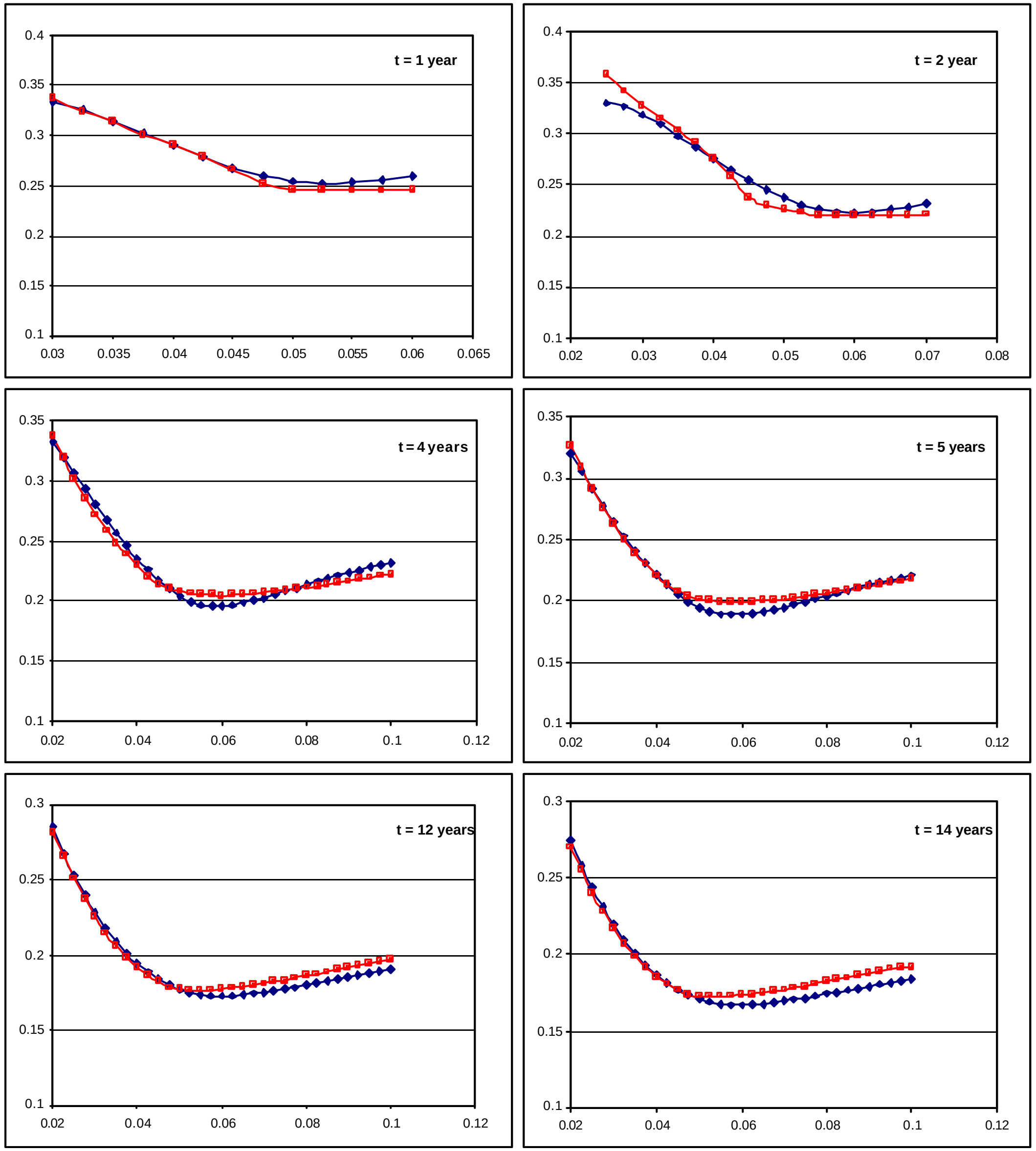
What does the stochastic-volatility look like?



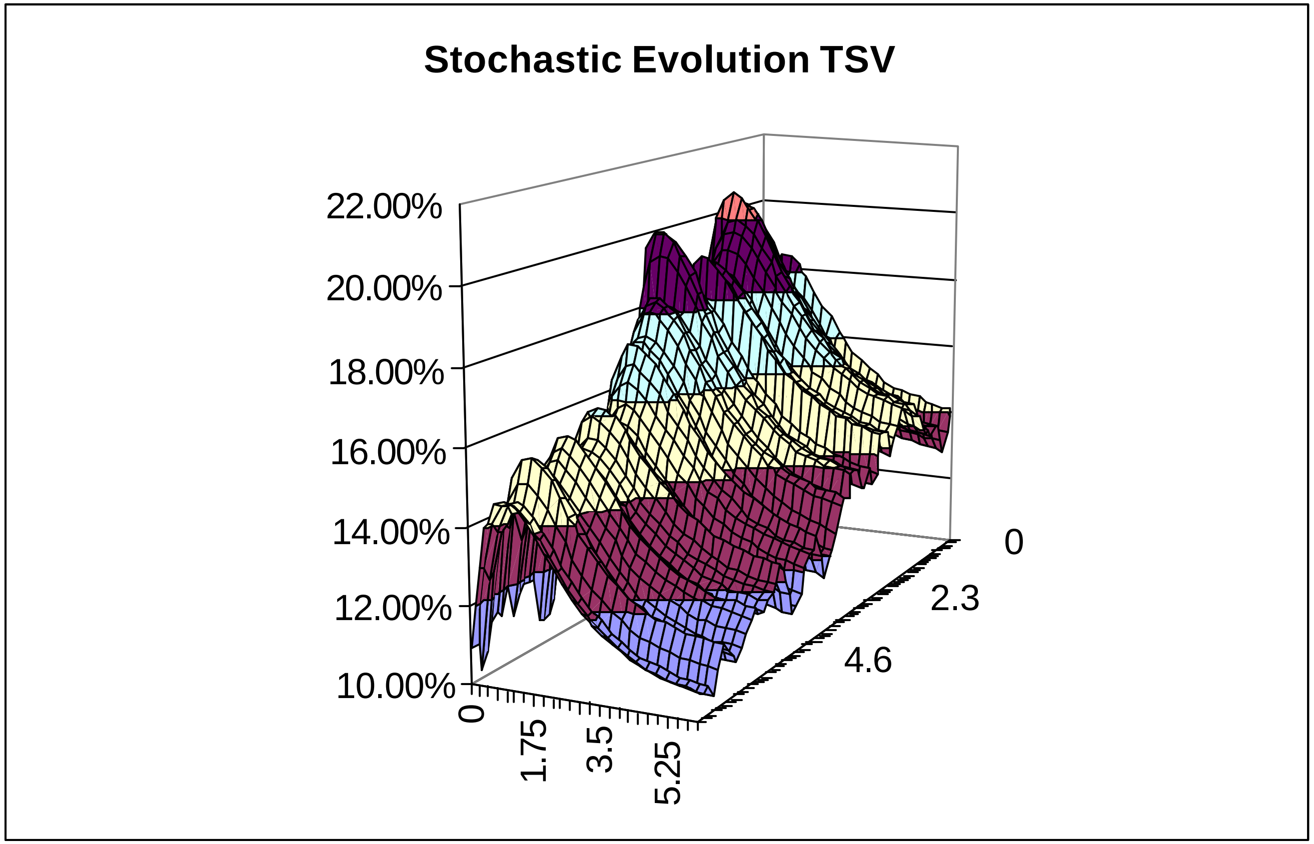

b stochastic
22%
21%
20%
19%
18%
17%
16%
15%
14%
0
2
4
6
8


a,b,c and d stochastic
23%
21%
19%
17%
15%
13%
11%
0
2
4
6
8

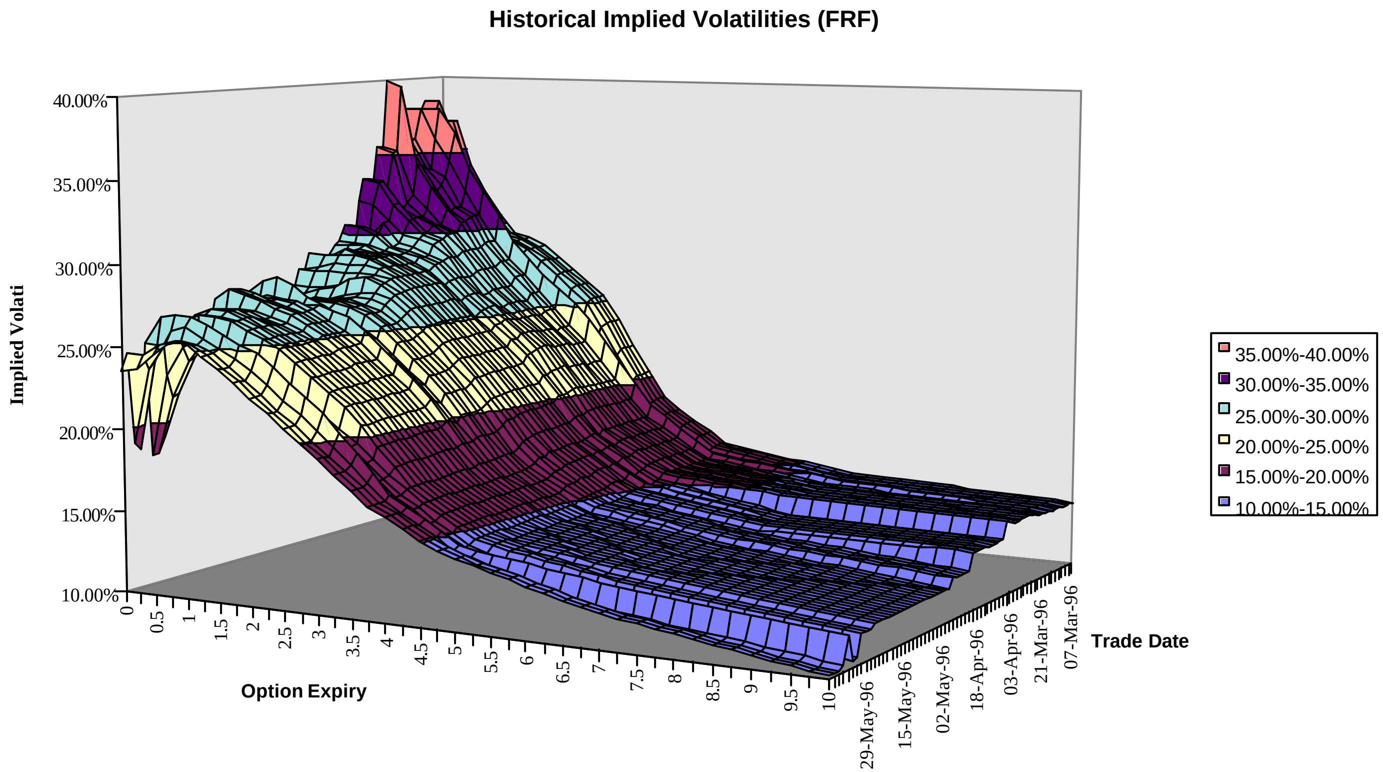
How well does it work? Fitting…



How well does it work? Theory…



… and Reality


Historical Implied Volatilities (FRF)
40.00%
35.00%
30.00%
25.00%
20.00%
15.00%
10.00%
Implied Volati
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
5
5.5
6
6.5
7
7.5
8
8.5
9
9.5
10
Option Expiry
29-May-96
15-May-96
02-May-96
18-Apr-96
03-Apr-96
21-Mar-96
07-Mar-96
Trade Date
35.00%-40.00%
30.00%-35.00%
25.00%-30.00%
20.00%-25.00%
15.00%-20.00%
10.00%-15.00%

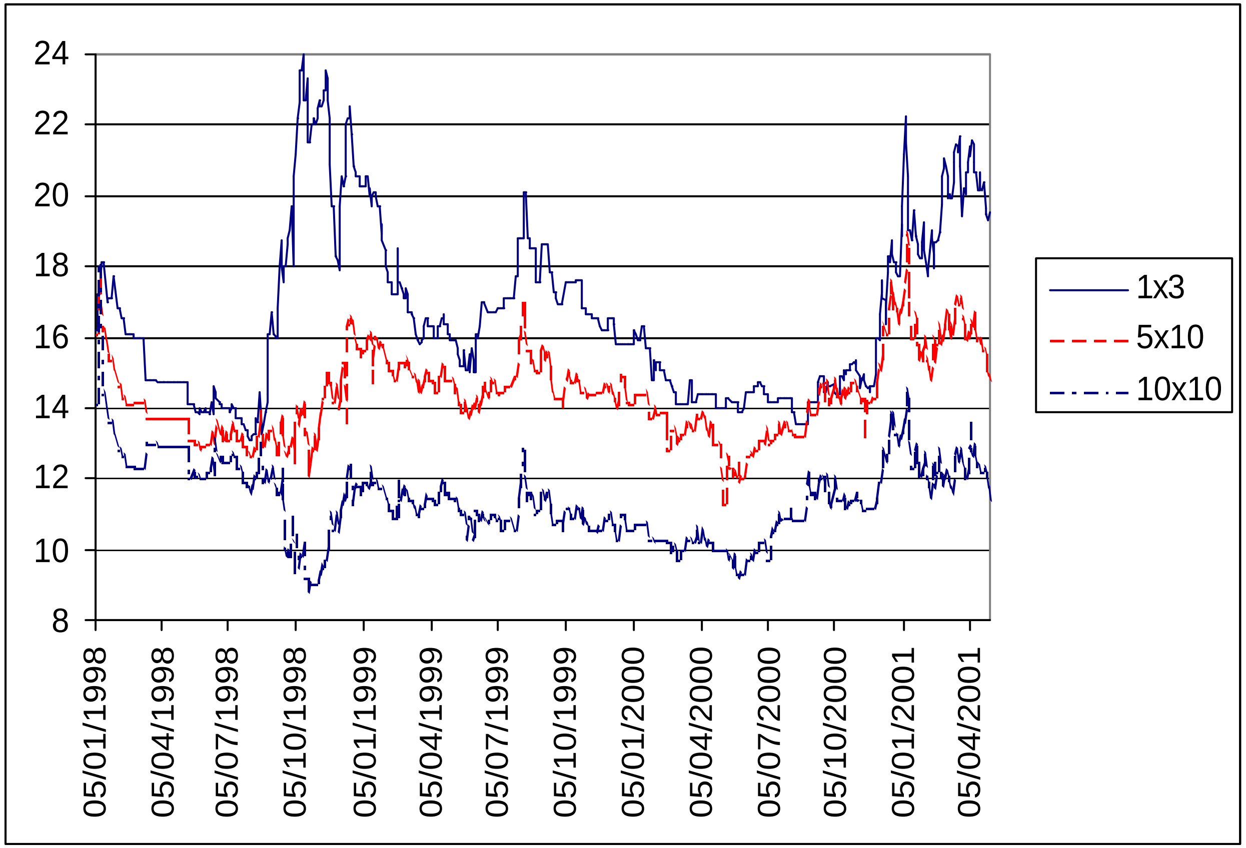
Am Important Feature

- Coefficients time-independent
- Forward-rate corrections necessary to price caplets exactly are very small

➔The statistical properties of the caplet and swaption surfaces are time-homogenous

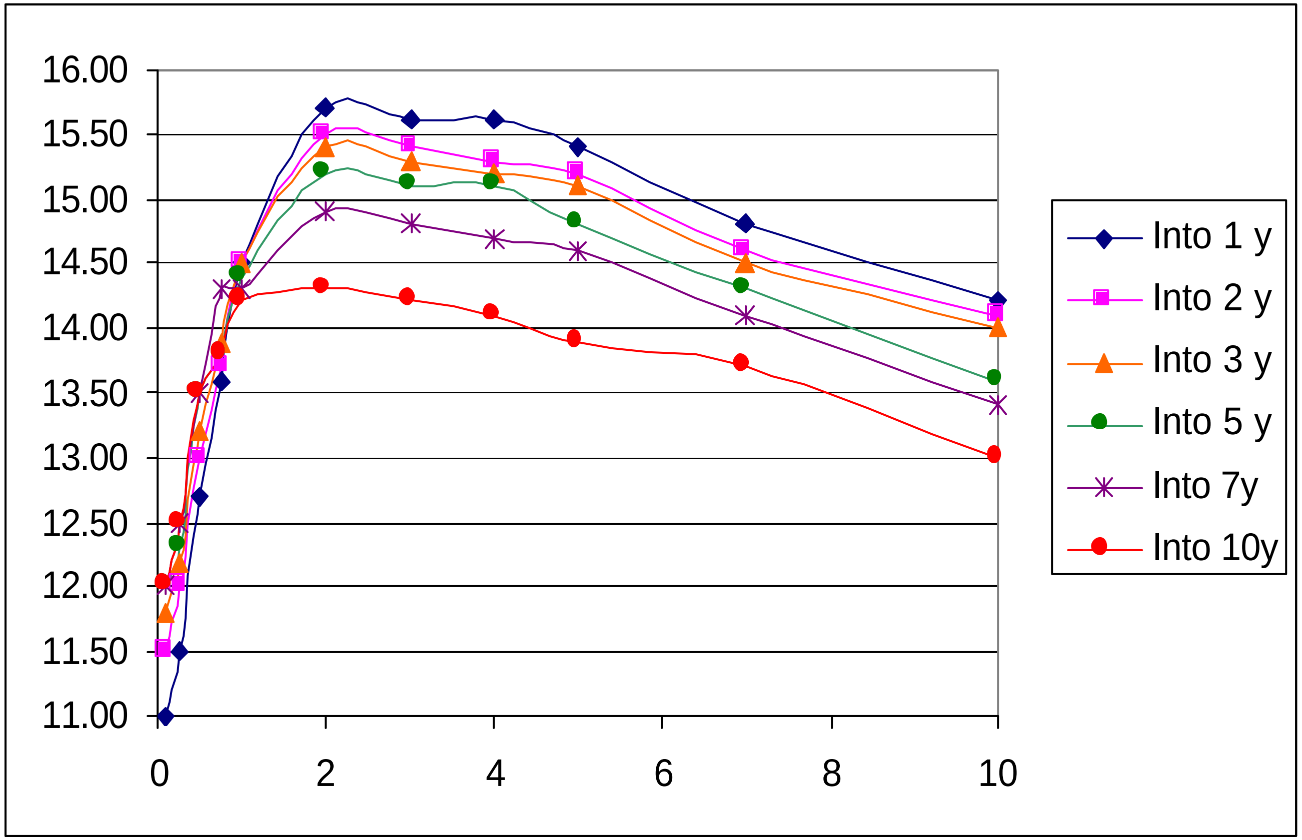
➔The Future looks like the present (and the recent past)

The problem

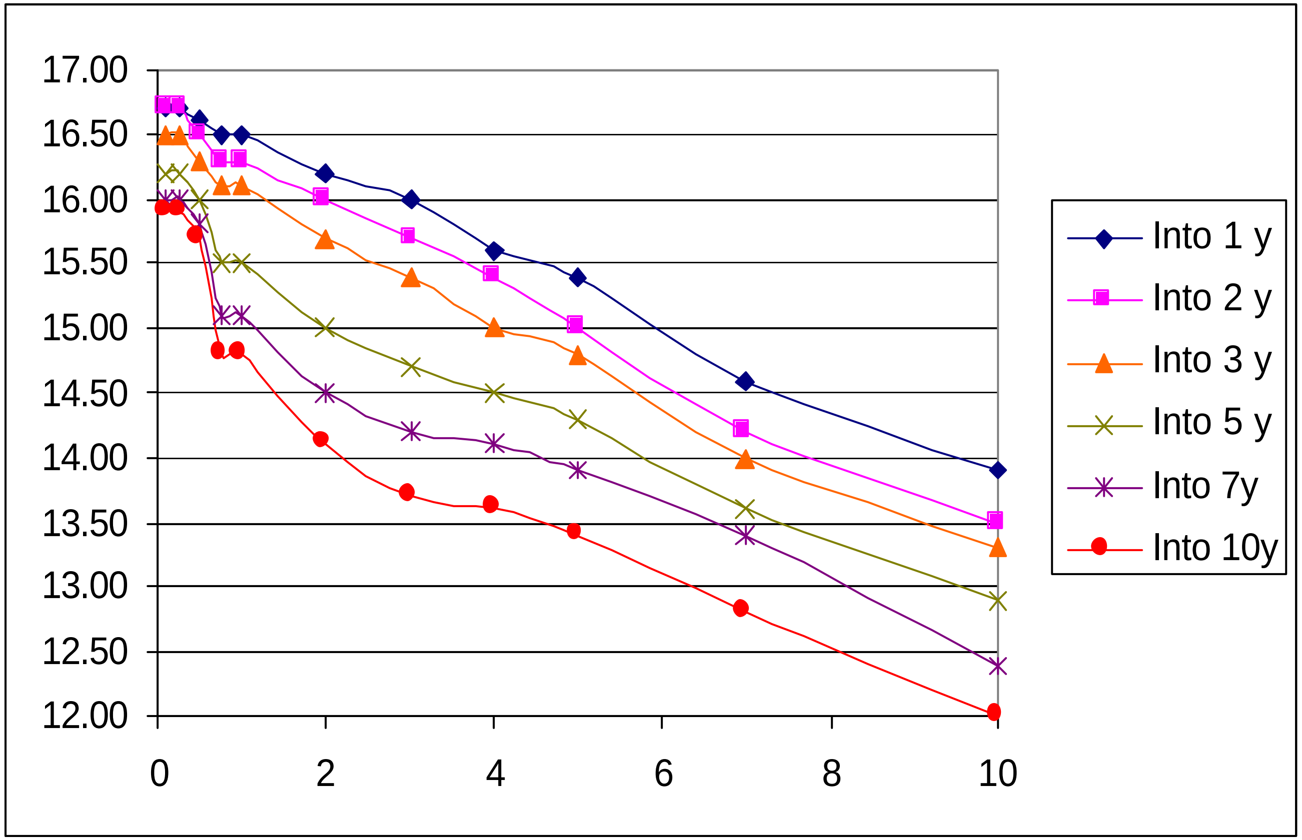


24
22
20
18
16
14
12
10
8
05/01/1998
05/04/1998
05/07/1998
05/10/1998
05/01/1999
05/04/1999
05/07/1999
05/10/1999
05/01/2000
05/04/2000
05/07/2000
05/10/2000
05/01/2001
05/04/2001
1x3
5x10
10x10

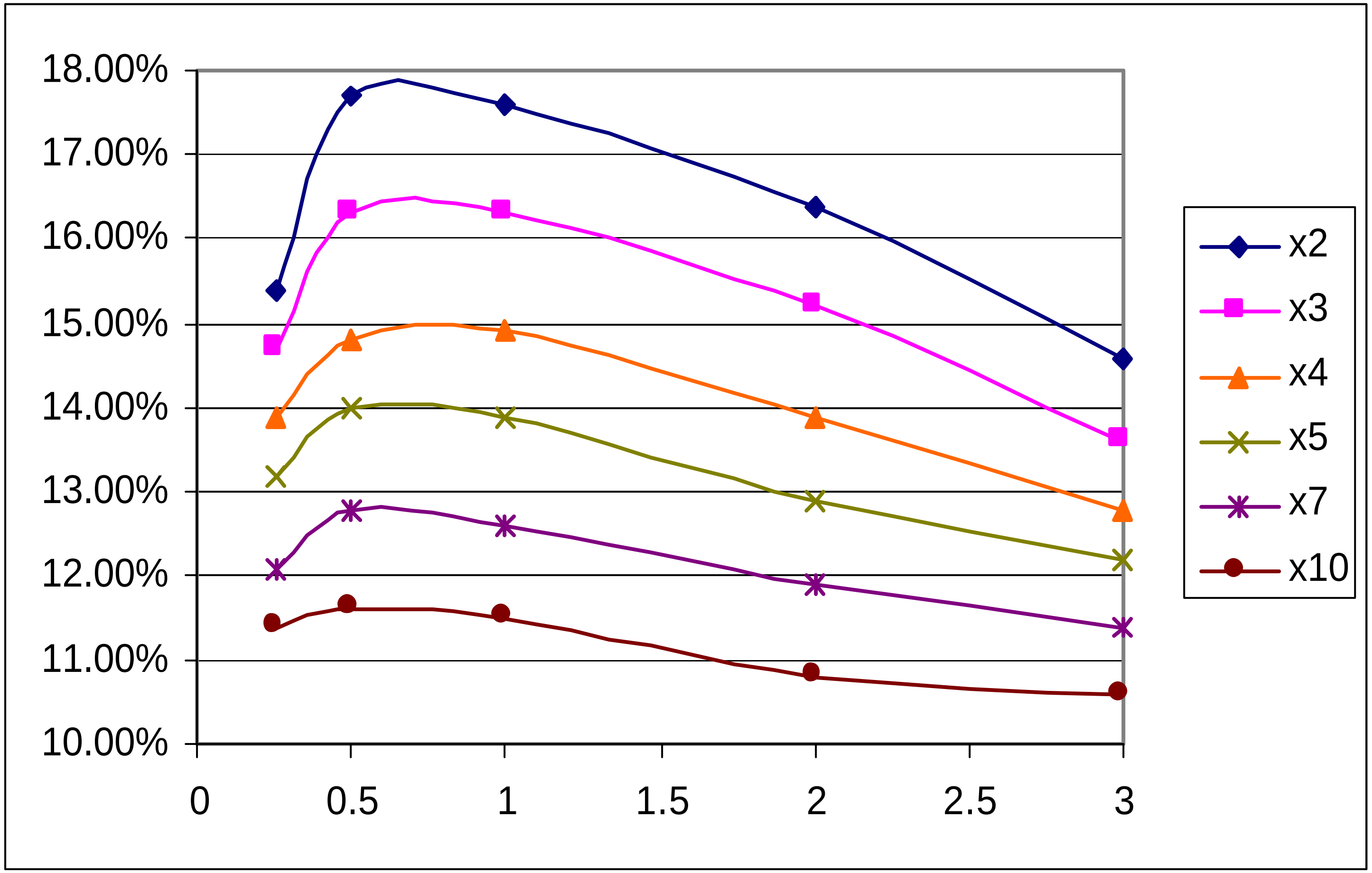
Normal pattern for US $



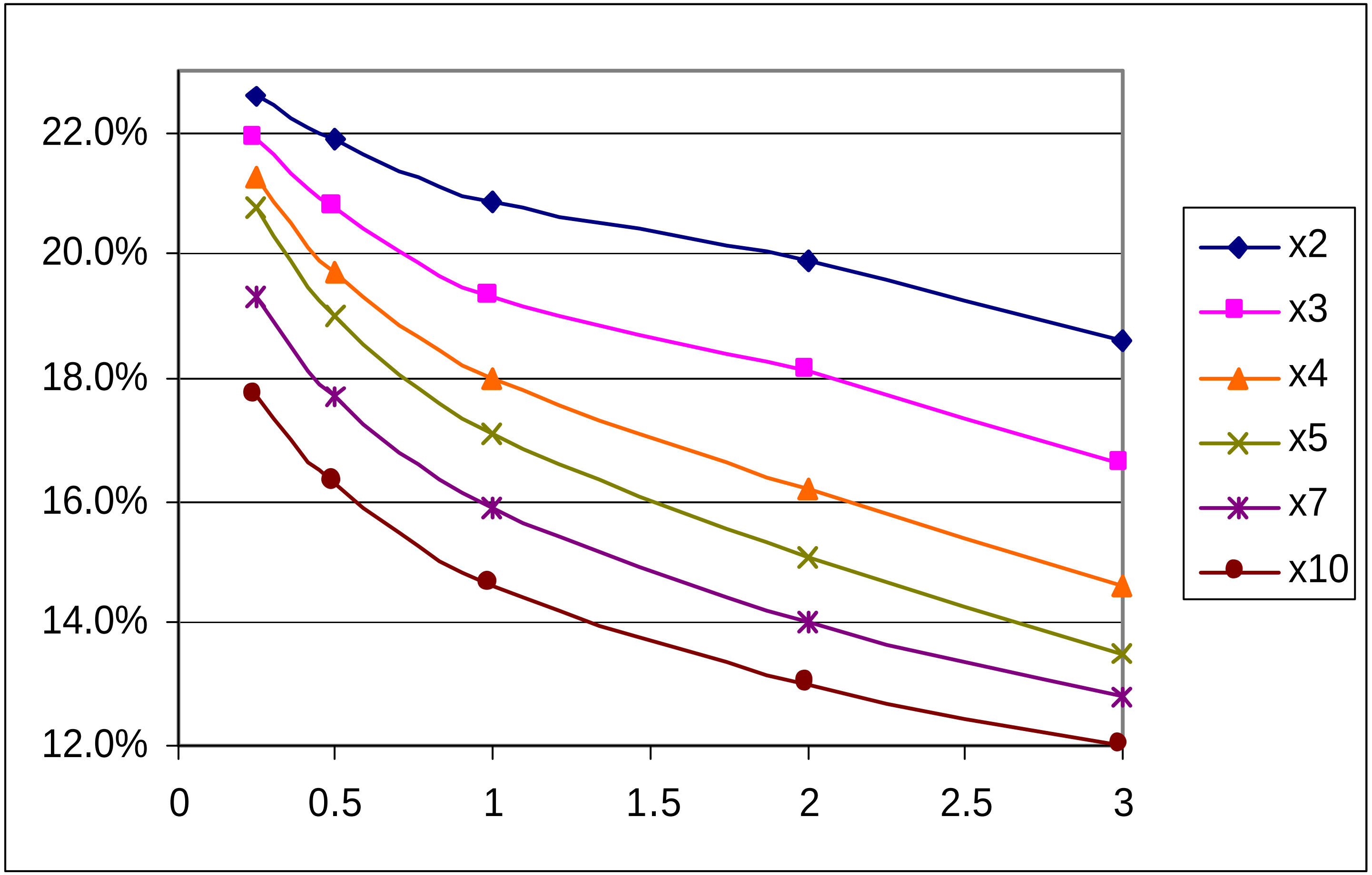
Excited pattern for US $



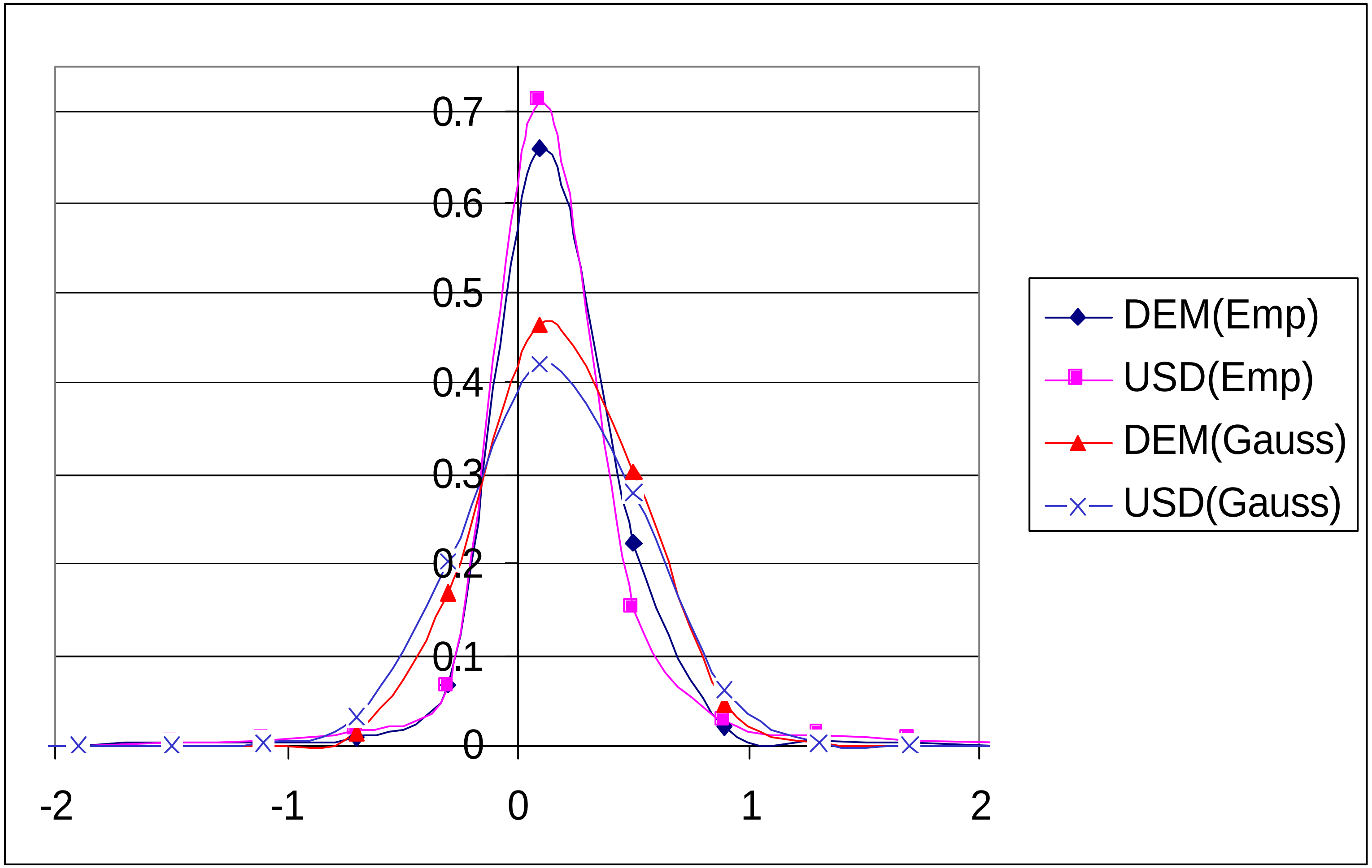
Normal pattern EUR/DEM



Excited pattern EUR/DEM



Density of changes in implied volatilities



The proposed solution

1. Choose a simple criterion to determine whether the swaption matrix is currently in the normal or excited state
2. The instantaneous volatility function for each forward rate can be described by either of these two functional forms:

$$\sigma_i^n(t,T_i) = [a_t^n + b_t^n(T-t)]\exp(-c_t^n(T-t)) + d_t^n$$

$$\sigma_i^x(t,T_i) = [a_t^x + b_t^x(T-t)]\exp(-c_t^x(T-t)) + d_t^x$$

The proposed solution [ctd]

3. All the coefficients {an,bn,cn, dn} and $\{a^x, b^x, c^x, d^x\}$ are stochastic, and follow the same Ornstein-Uhlenbeck process described in the original work. Their processes are all uncorrelated with the forward rates

4. The transition of the instantaneous volatility from the normal to the excited state occurs with frequency $?_{n?\ x}$, and the transition from the excited state to the normal state with frequency $?_{x?\ n}$. Notice that both frequencies are risk-adjusted and not real-world frequencies; and that $?_{n?\ x}+?_{x?\ n}?1$

The proposed solution [ctd]

5. Since the same assumption of independence between the volatility processes and the forward rate processes is enforced, once again along each volatility path the problem is exactly equivalent to the deterministic case, apart from the fact that, at random times, the coefficients would switch from one state to the other

Features of the approach

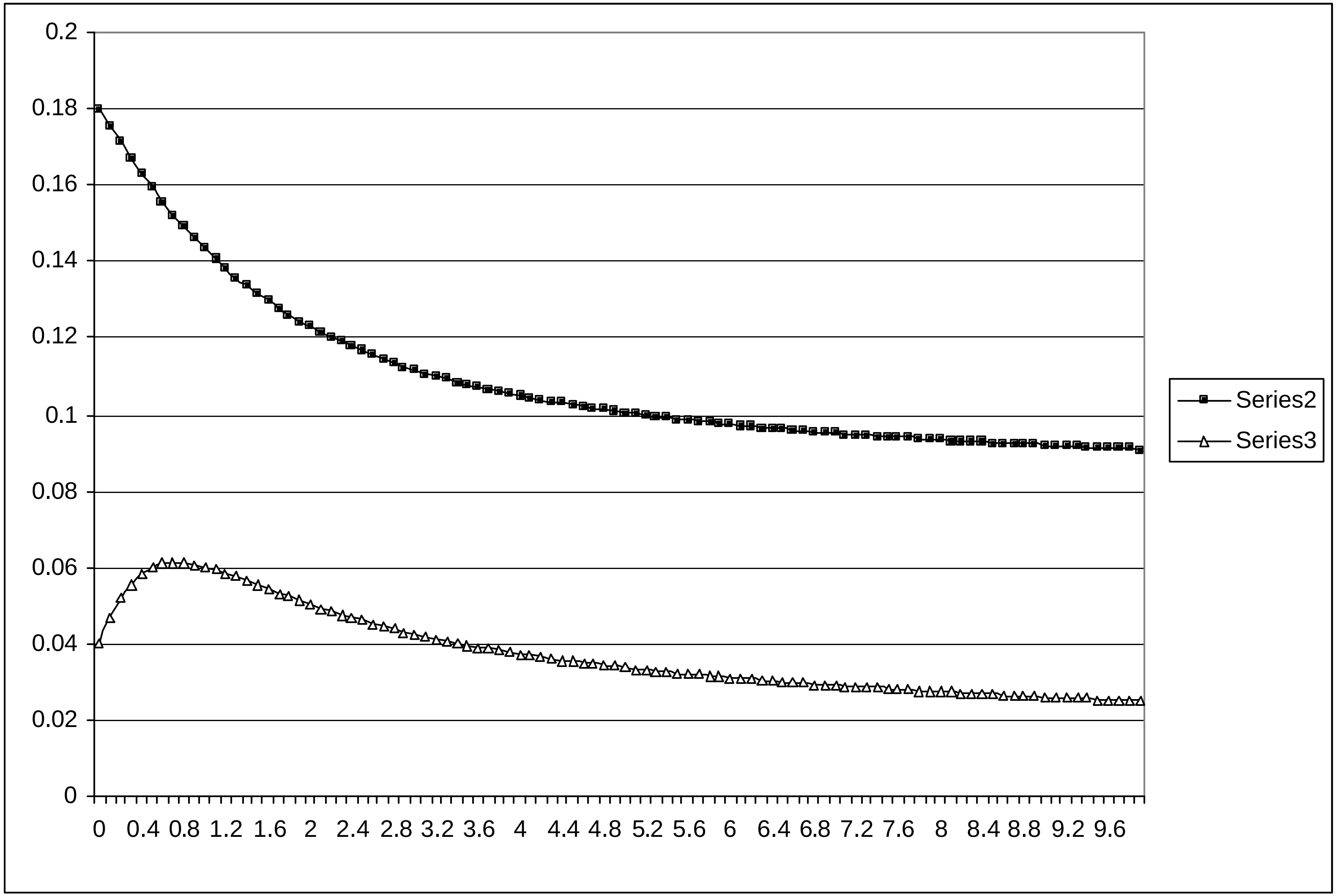
This procedure successfully capture the most significant qualitative features highlighted by the empirical study. In particular

- the transition between states and the reversion to the two basic (normal and excited modes) is built into the model;
- the implied volatility distribution displays fatter tails than in the pure-diffusion case; and
- the explanatory power of the first eigenvector is lower than in the simple-diffusion case.

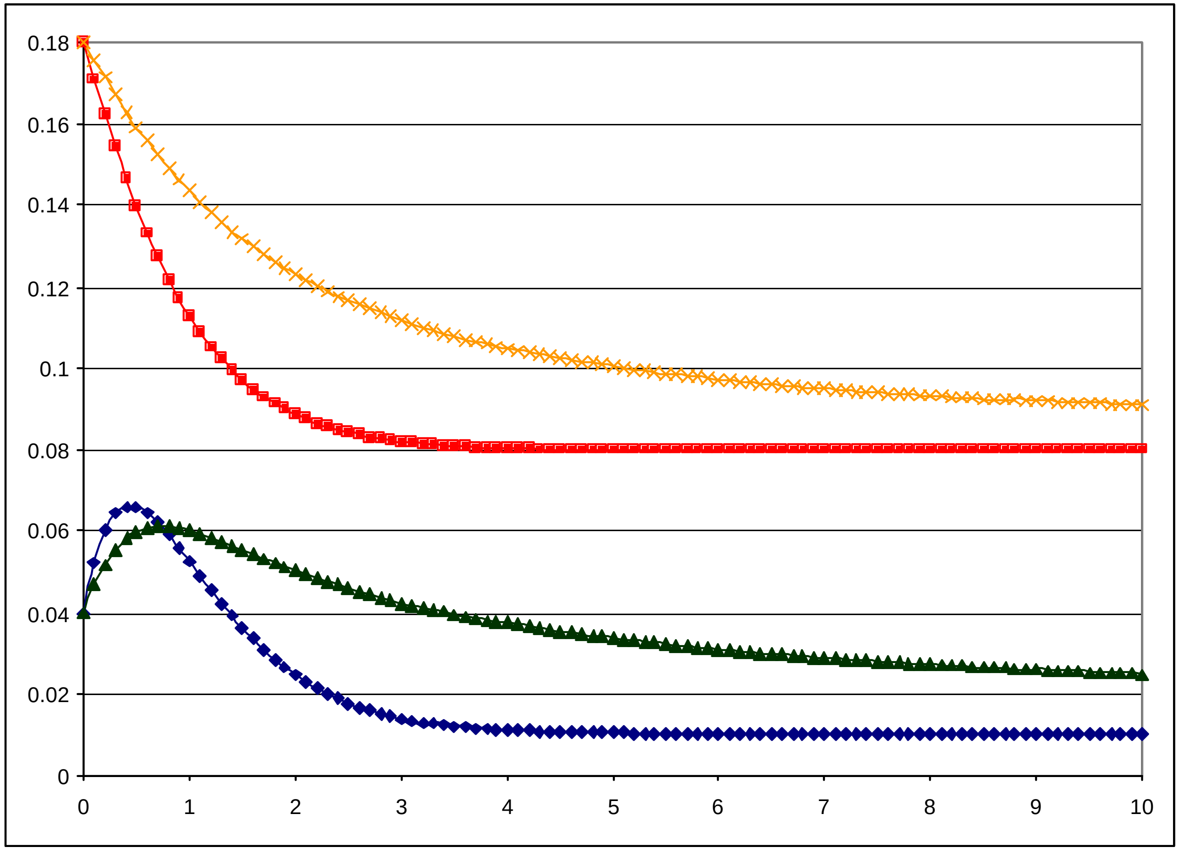
Features of the approach

- All the coefficients are time-independent – the term structure of volatilities is time-homogeneous
- Structural features of the swaption matrix do not change over time
- The future looks (statistically) like the past

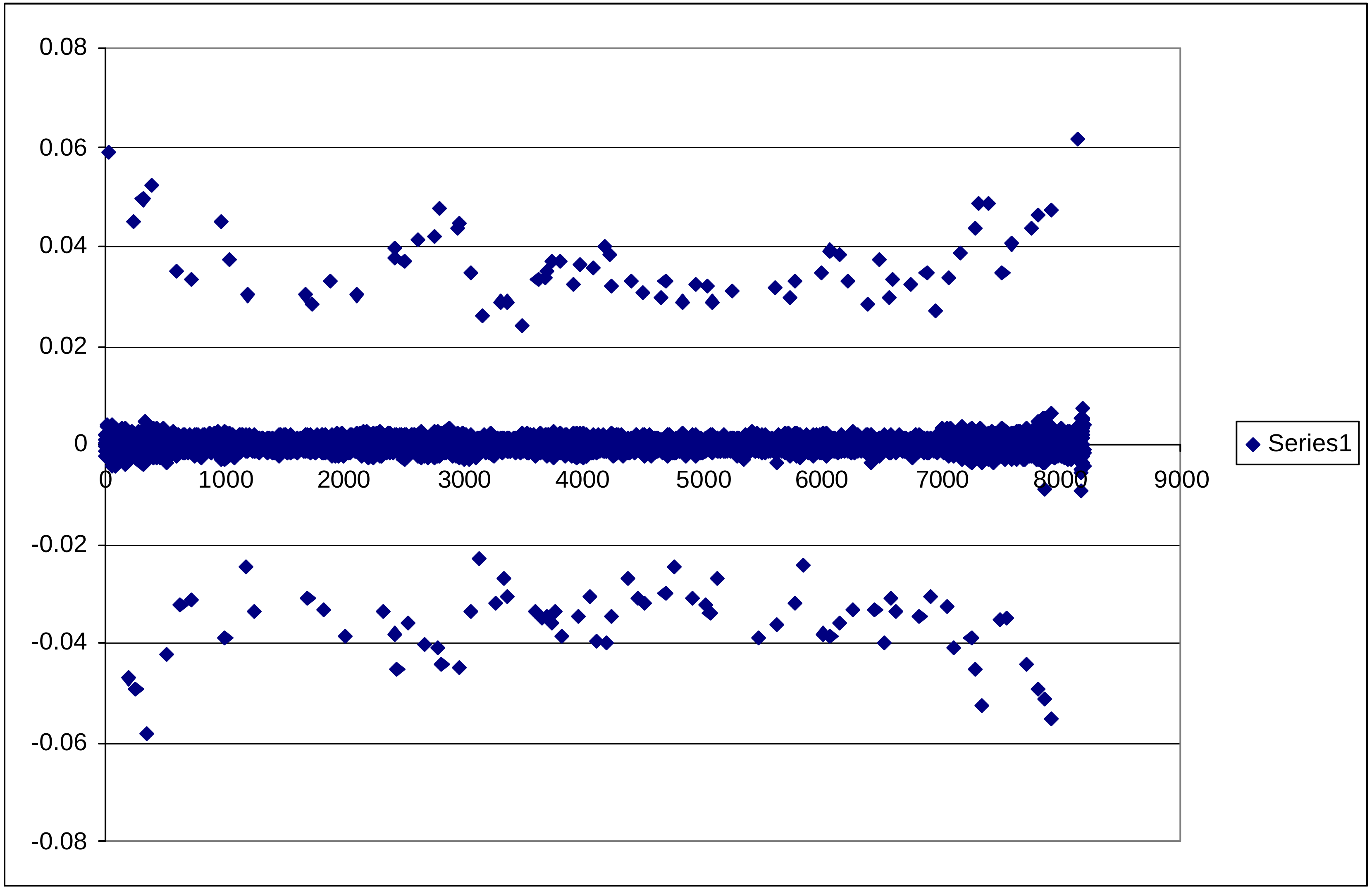
The two-state volatilities



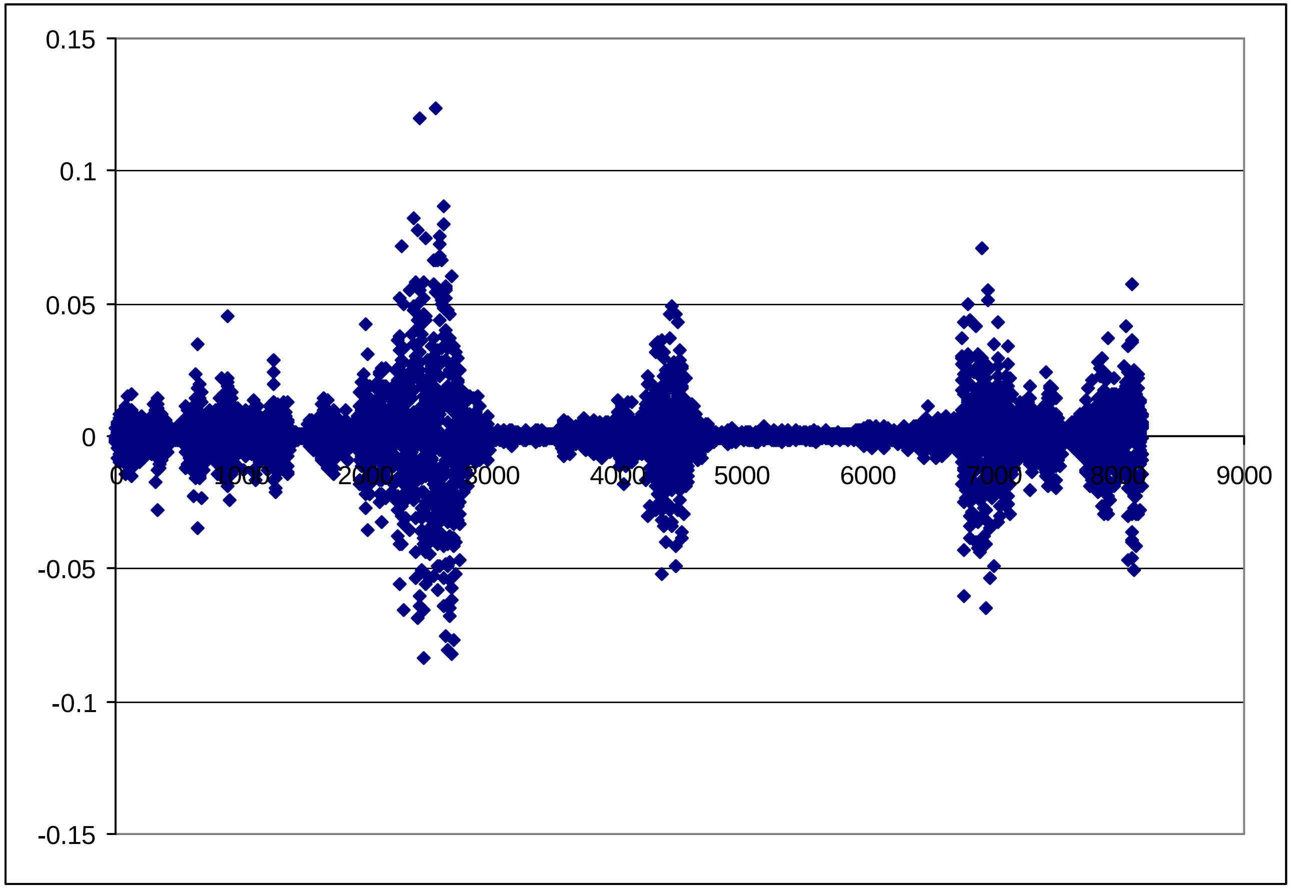
Another example



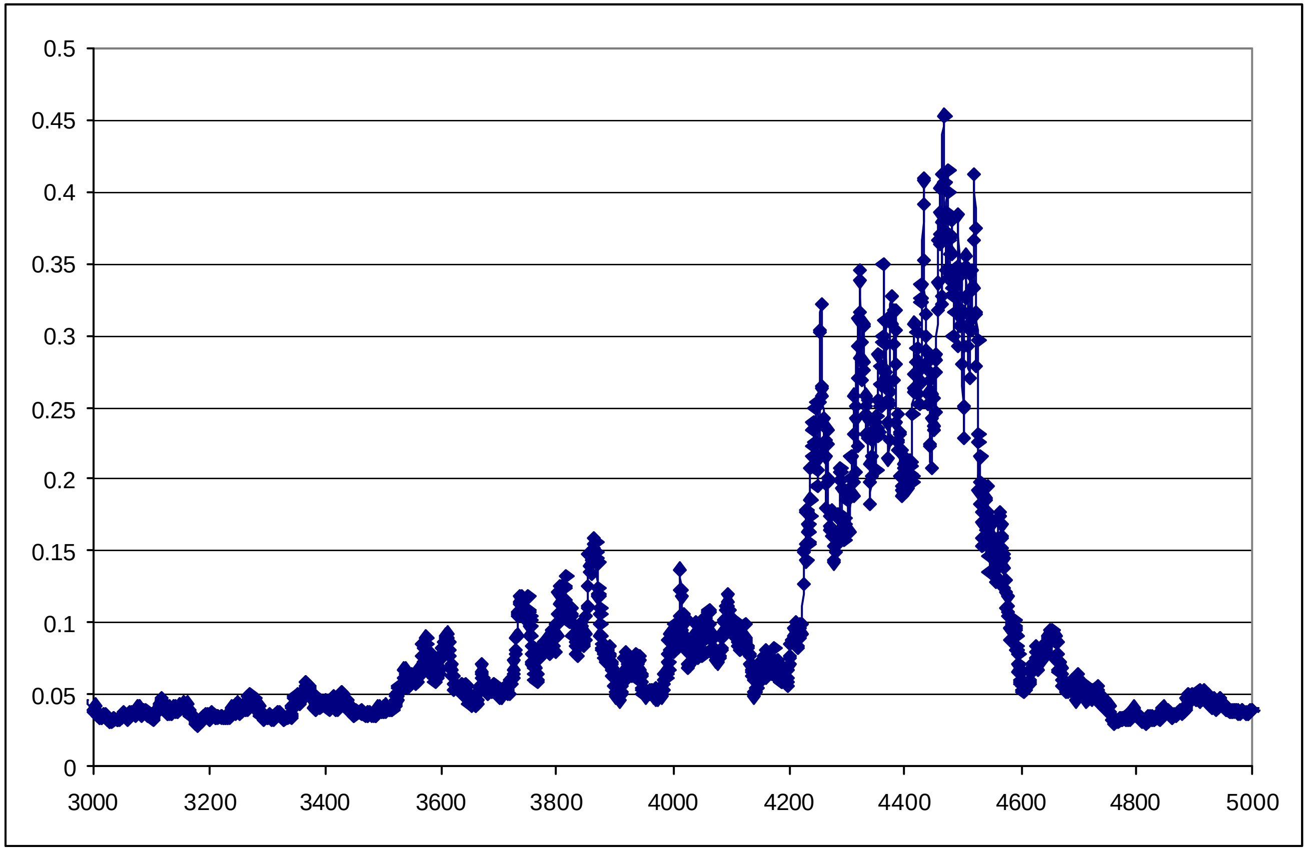
The Instantaneous Volatility Path



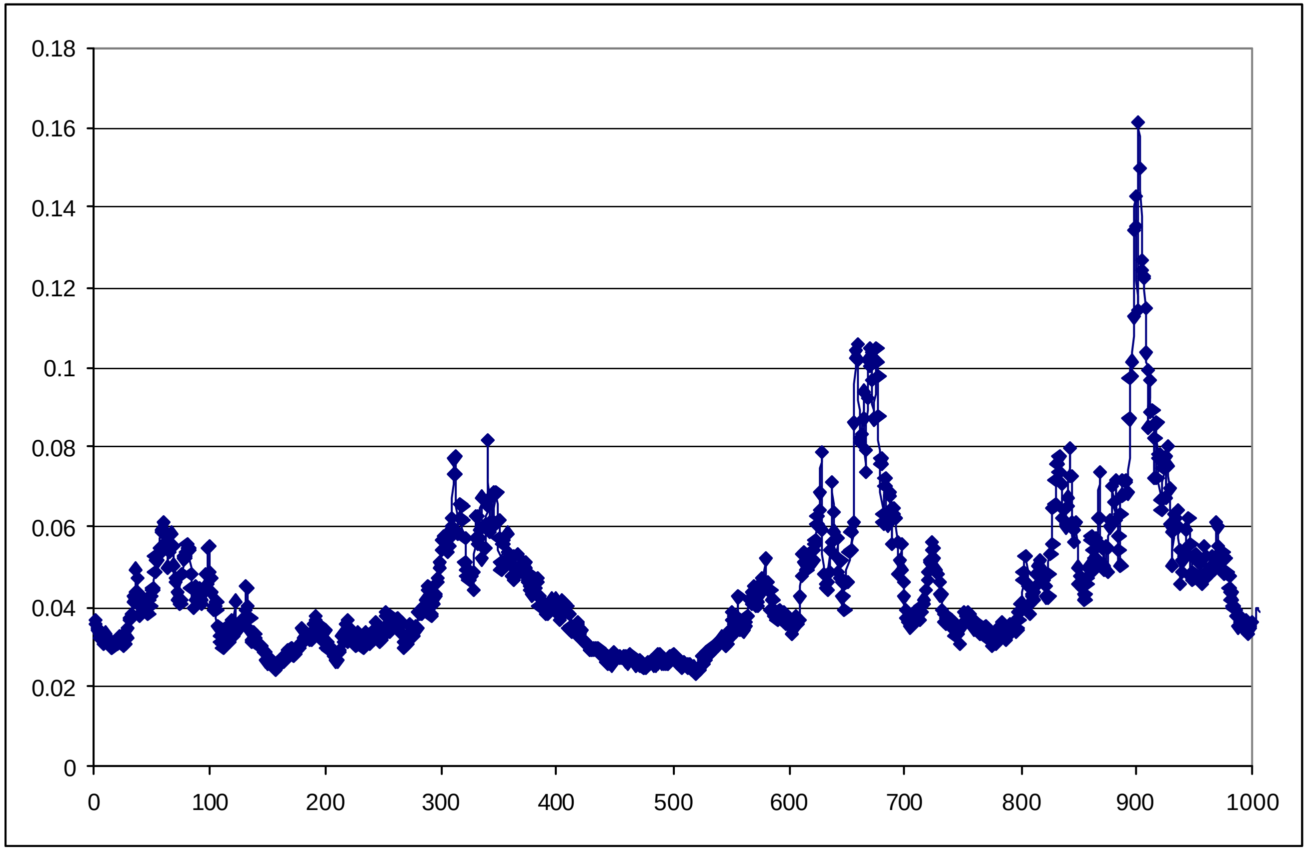
The Instantaneous Volatility Path



The Instantaneous Volatility Path



The Instantaneous Volatility Path



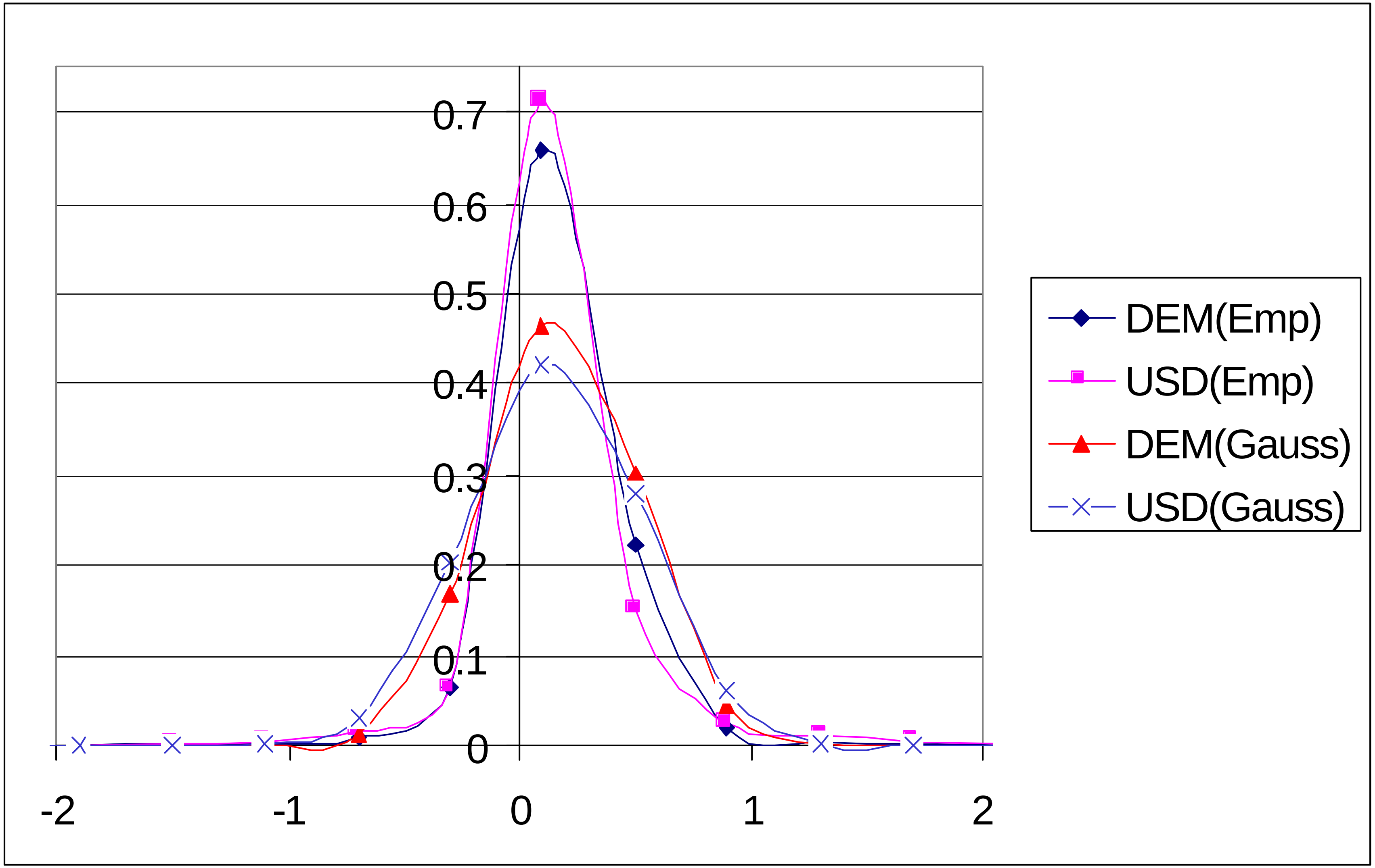
The investigation methodology

1. Choose the volatility parameters
2. Move the yield curve and the volatility forward by one day
3. Calculate the prices of the swaptions
4. Repeat from 2. and collect the changes
5. Construct eigenvalues and eigenvectors

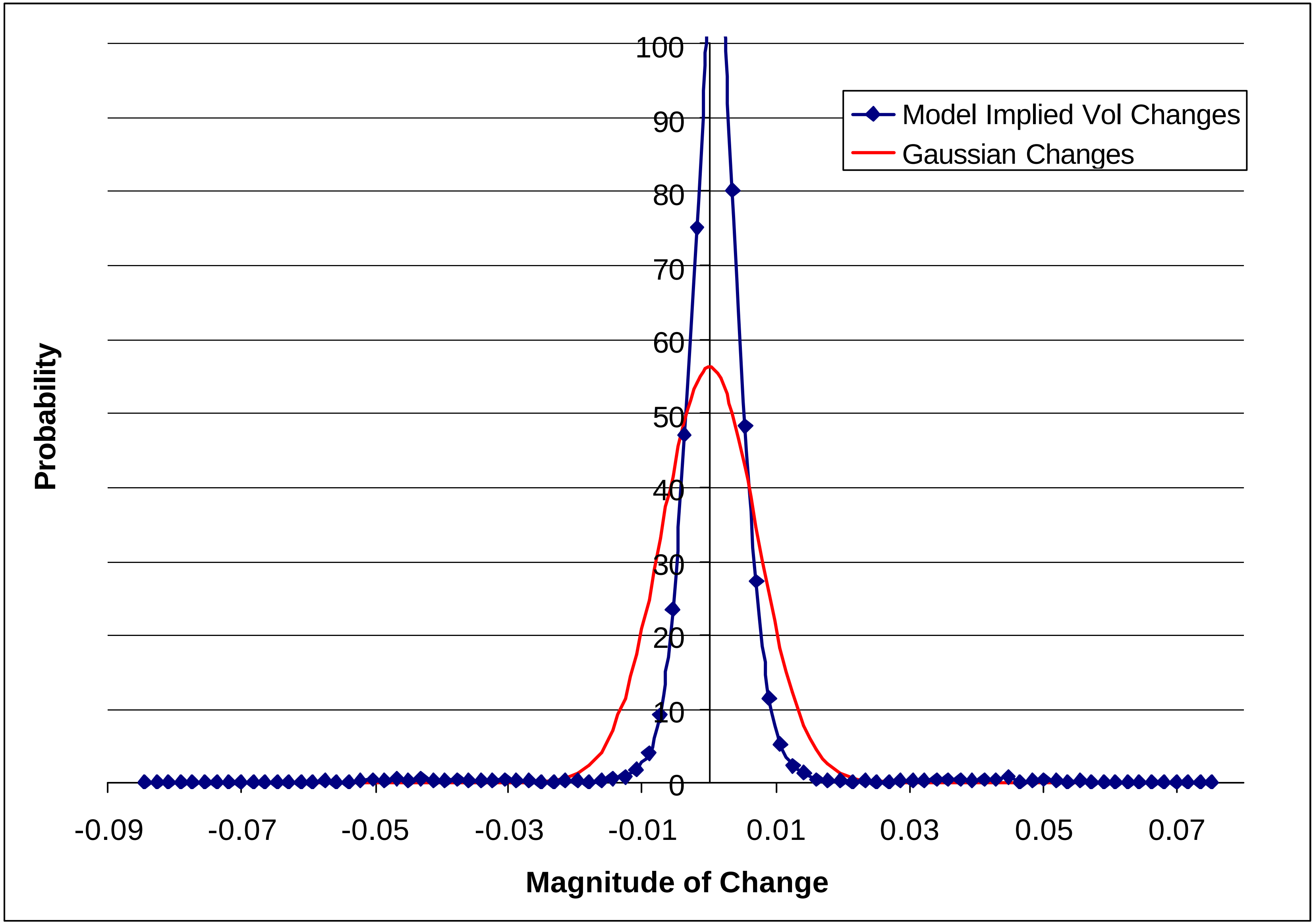
Why looking at the eigenvalues?

- Naïve comparison between real-world and risk-adjusted world is not warranted
- In the limit the eigenvectors/eigenvalues are unchanged in a drift transformation

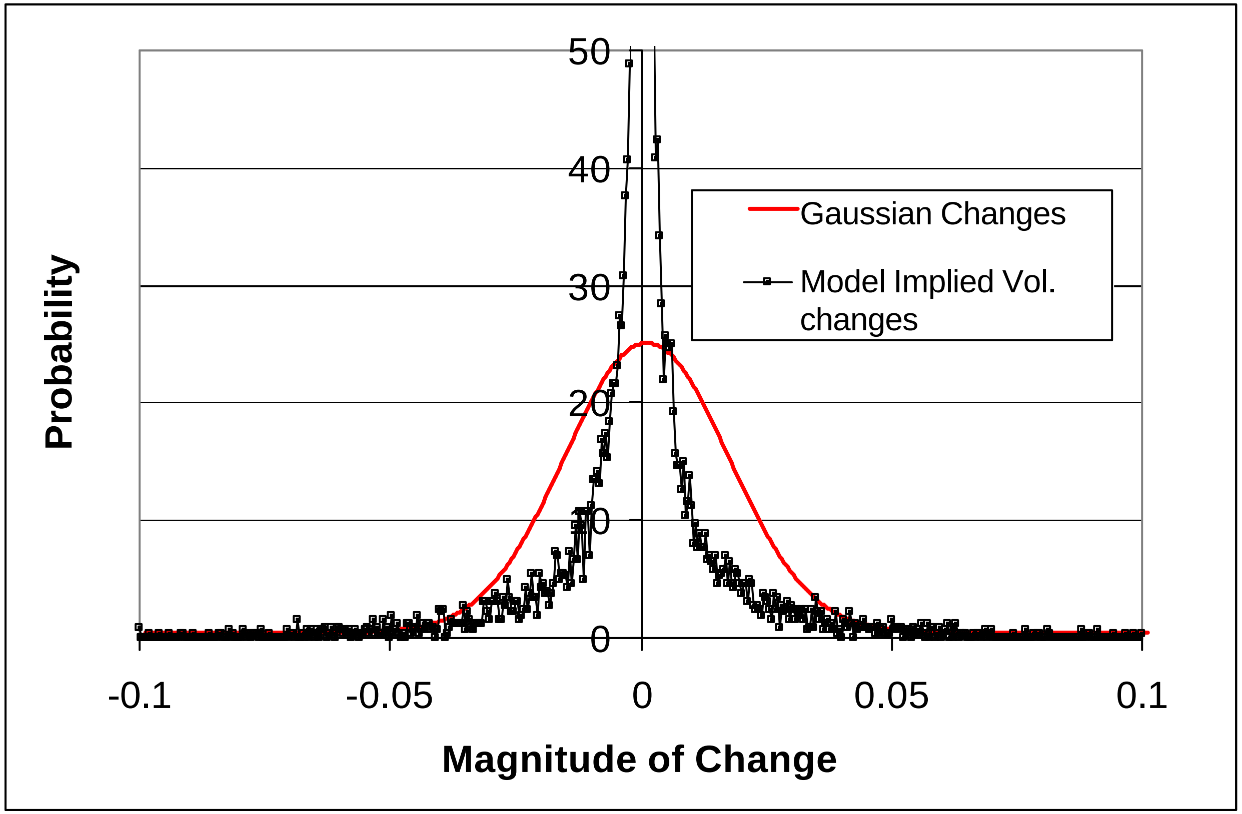
The old approach…



The model volatility changes: new approach



The model volatility changes:new approach



Model and Real Skew

SKEW	Real	Model
1x2	0.767421	0.202036
1x5	0.529616	0.386044
1x10	0.868207	0.296187
3x3	1.421207	0.352619
3x5	0.443754	0.327331

Model and Real Skew

SKEW	Real	Real(I)	Real(II)	Model
1x2	0.767421	1.056023	-0.03201	0.202036
1x5	0.529616	0.515964	0.322293	0.386044
1x10	0.868207	1.260718	0.501672	0.296187
3x3	1.421207	1.36261	0.294108	0.352619
3x5	0.443754	0.375826	0.140795	0.327331

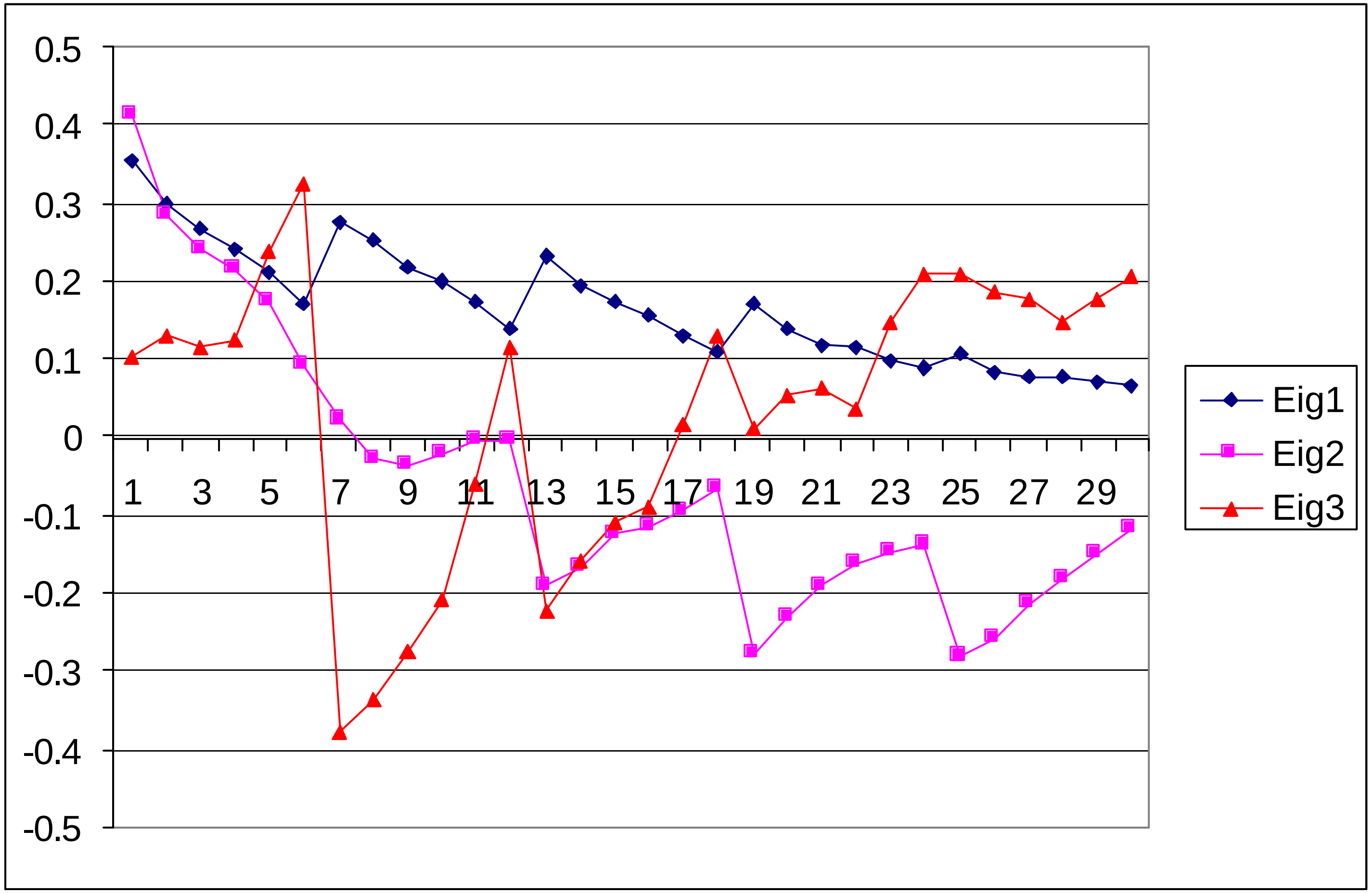
Model and Real Kurtosis

KURT	Real	Model
1x2	14.3935	11.43534
1x5	10.10832	15.01118
1x10	20.11422	17.20576
3x3	15.93261	17.2059
3x5	11.70314	21.18686

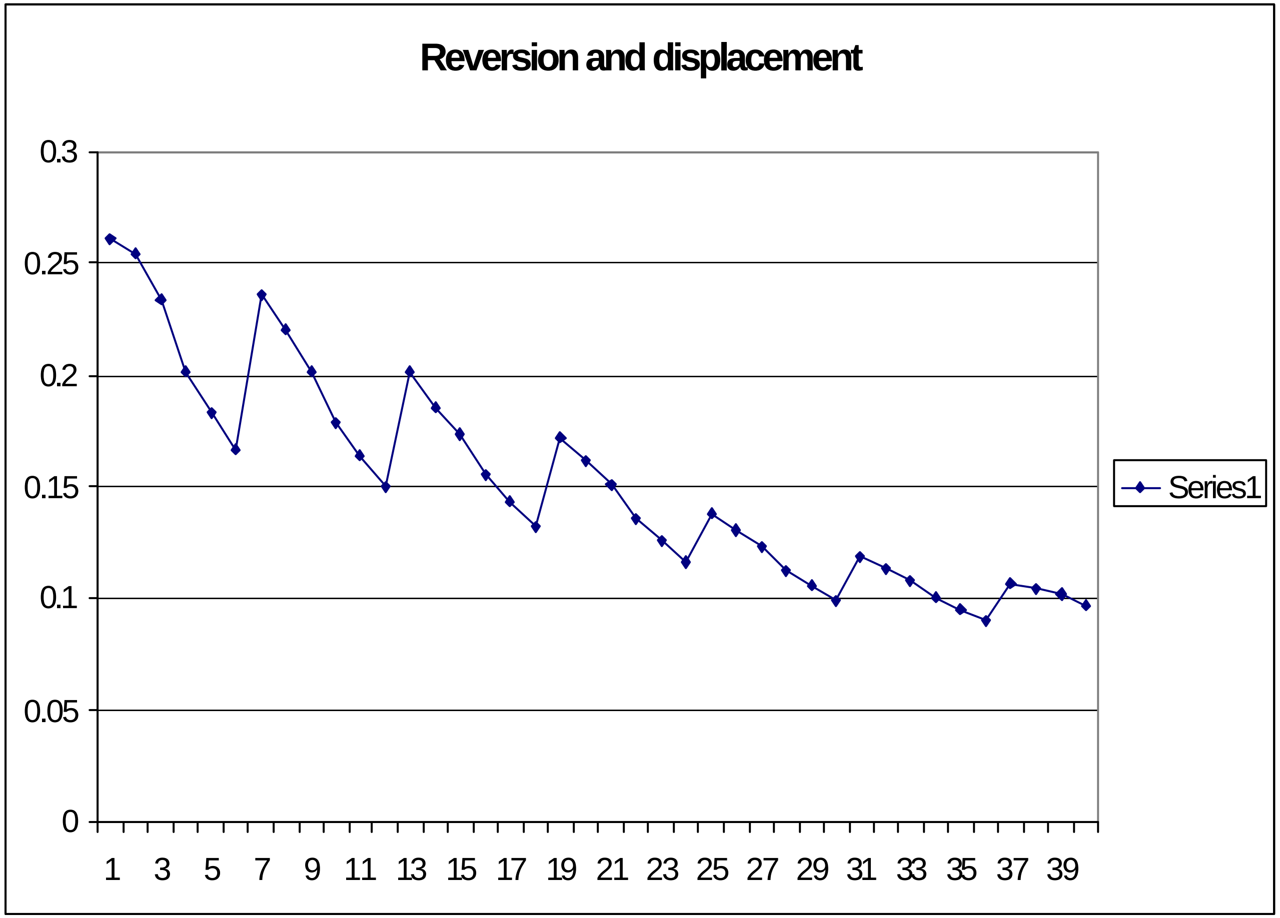
Model and Real Kurtosis

KURT	Real	Real(I)	Real(II)	Model
1x2	14.3935	16.09731	4.821323	11.43534
1x5	10.10832	10.27761	2.640147	15.01118
1x10	20.11422	11.99224	28.02998	17.20576
3x3	15.93261	12.06961	4.364113	17.2059
3x5	11.70314	8.457682	3.049932	21.18686

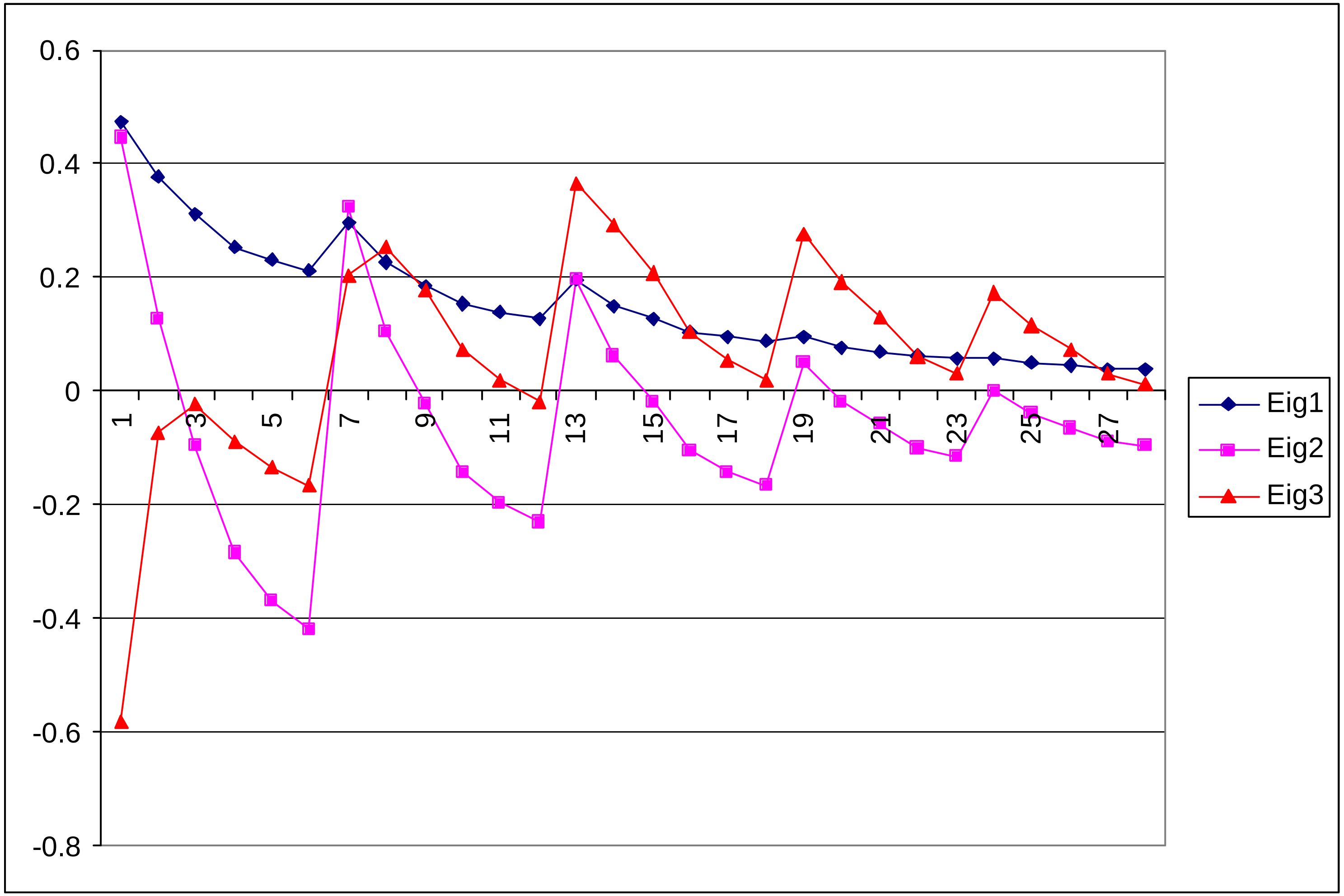
Eigenvectors: real data



Eigenvectors:old approach



Eigenvectors: new approach

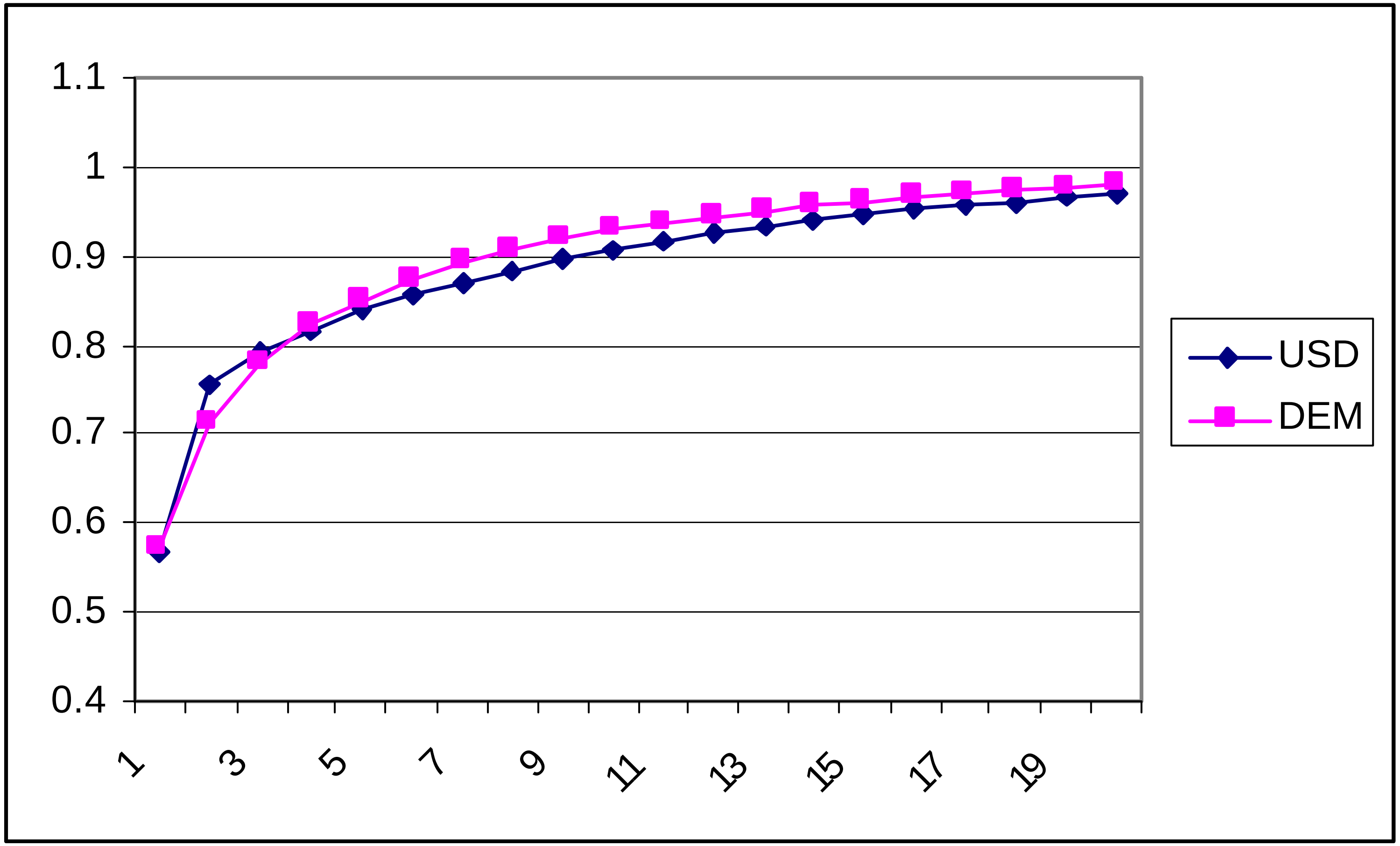


Why do we care?

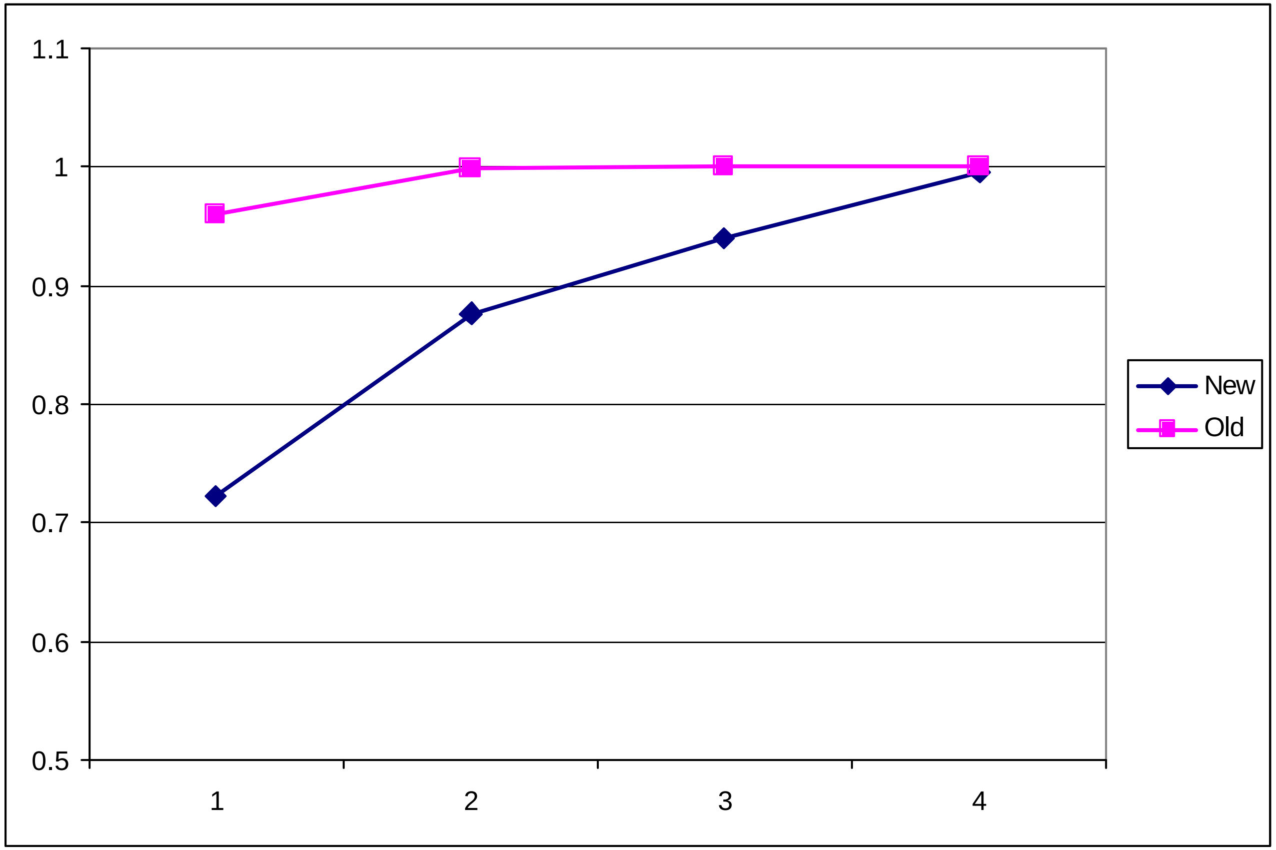
What does a stochastic-volatility model really buy me?

- Pricing with a deterministic-volatility LMM
- The information from ‘gamma-vega’
  - **A stochastic-volatility model automatically ‘knows’ about gamma vega**

Eigenvalues: real data



Eigenvalues: model data



“The fitting is just as good with a ‘simple’ SVLMM. Do I really care?”

- Perfect fitting to caplets can be obtained with an infinity of models
- All these models produce different prices
- Future shapes of the implied volatility surface will determine the prices will I will transact my future re-hedges
- Implicitly, all models assign a cost to these future re-hedges.

What constitutes a successful model?

- Risk-neutral pricing versus actuarial pricing: in the case of incomplete markets, perfect replication is impossible, and only partial hedging can be achieved
- If perfect replication is possible, I do not care about the future (real-world) realization of the underlying variable: I can ‘lock in’ the prices implied by the market
- If hedging is totally impossible, I must make my price on an actuarial basis. Statistical knowledge of the real-world evolution becomes all-important

What constitutes a successful model?

- Reality is in between: we should hedge as much as possible, but recognize that our payoff replication will be imperfect
- The greater the difficulty in hedging, the more important the ability for a model to describe the real world behaviour of the unhedgeable quantities

Conclusions

The new approach:

- Lends itself to rapid evaluation of path-dependent products
- Allows rapid calibration to caplets and to the European swaption matrix
- Allows financially desirable exact calibration to the co-terminal swaptions that underlie a given Bermudan
- Recovers the shape of the eigenvectors in a very satisfactory way
- Gives much better ratio of eigenvalues

Conclusions [ctd]

- Accounts for the sudden transitions observed in the market between volatility regimes
- Produces a very reasonable distribution of implied volatility changes
- Conditionally on one volatility path being realized, the problem is reduced to a deterministic volatility LMM, and the approach can therefore be used for practical pricing.