Risk managing long-dated smile risk with SABR formula

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Abstract

In this paper[1], we show that the sensitivities to the SABR parameters can be materially wrong when the SABR formula is used, in particular for long expiries and in high volatility environments. For example, we obtain positive sensitivities to the spot-vol correlation parameter (ρ) for low strike options, which is the opposite of what we expect from the SABR model. We discuss possible solutions.

1 Introduction

The SABR stochastic volatility model is very popular among practitioners, especially within IR and FX, due to the availability of an asymptotic approximation for Black-Scholes (BS) implied volatilities [1]. It is well known however that this formula can be inaccurate, in particular for high volatilities, low strikes and long expiries.

The first implication of such an inaccuracy is a discrepancy between the true model prices of vanilla options (calculated, for example, by Monte Carlo) and those obtained using the SABR formula. This issue is often overlooked, to the extent that the SABR model is used as a smile interpolation/extrapolation tool rather than a pricing model for exotics.

A second implication of Sabr formula's inaccuracy is that the asset distribution it implies is not guaranteed to be arbitrage-free, i.e. the probability density function implied from vanilla option prices computed using Sabr formula may be negative in some regions, typically for low values of the underlying.

A third implication of the SABR's formula inaccuracy, which is usually overlooked and which is the topic of this note, is that when the formula is not accurate (e.g. for long expiries in high volatility environments) sensitivities to Sabr parameters may not be adequate tools for risk management, since in this case Sabr parameters can cease to be financially meaningful quantities. We find that this is the case in the current market for long-dated swaptions in USD, where the inaccuracy of Sabr formula is now larger than it has been historically as a result of the current high lognormal volatilities, which in turn are the result of the low levels of interest rates. We find that in this market the sensitivity to Sabr correlation for low strike options can be positive for long expiries, which is the opposite of what we expect from Sabr model and of what is observed for shorter expiries.

This note is organised as follows: section 2 summarises the SABR model's equations and formulas; section 3 discusses the meaning of the SABR model's parameters; in section 4 we show that the SABR formula can generate sensitivities to the SABR parameters which

[1] I would like to thank Dherminder Kainth for his helpful comments on an earlier draft.

are the opposite of those of the Sabr model, and which are inadequate for risk management; finally, section 5 discusses some possible solutions to the problems highlighted in section 4.

2 Sabr model

The SABR model [1] assumes the following dynamics for the underlying S:

$$\mathrm{d}S(t) = \sigma(t)\, S(t)^{\beta}\, \mathrm{d}W(t), \tag{1}$$

$$\frac{\mathrm{d}\sigma(t)}{\sigma(t)} = \nu\, \mathrm{d}W'(t), \;\; \sigma(0) = \alpha, \tag{2}$$

where W and W' are correlated Brownian motions with correlation ρ.

An approximation for the BS implied volatility generated by this model has been obtained in [1]:

$$\Sigma(S,K,T) = I_0\,(1 + I_1\,T), \tag{3}$$

$$I_0 \triangleq \frac{\alpha}{(SK)^{(1-\beta)/2}\left\{1 + \frac{(1-\beta)^2}{24}\ln^2(S/K) + \frac{(1-\beta)^4}{1920}\ln^4(S/K)\right\}}\frac{z}{x(z)}, \tag{4}$$

$$I_1 \triangleq \frac{(1-\beta)^2}{24}\frac{\alpha^2}{(SK)^{(1-\beta)}} + \frac{1}{4}\frac{\rho\beta\nu\alpha}{(SK)^{(1-\beta)/2}} + \frac{2-3\rho^2}{24}\nu^2, \tag{5}$$

$$z \triangleq \frac{\nu}{\alpha}(SK)^{(1-\beta)/2}\,\ln(S/K), \tag{6}$$

$$x(z) \triangleq \ln\left(\frac{\sqrt{1-2\rho\,z+z^2}+z-\rho}{1-\rho}\right). \tag{7}$$

In [4] a more accurate approximation for the case $\beta \in (0,1)$ is presented, which uses (3), (5), replacing (4) with:

$$I_0 \triangleq \frac{\nu\,\ln(S/K)}{\ln\left(\frac{\sqrt{1-2\rho\,q+q^2}+q-\rho}{1-\rho}\right)}, \tag{8}$$

$$q \triangleq \frac{\nu}{\alpha}\frac{S^{1-\beta}-K^{1-\beta}}{1-\beta}. \tag{9}$$

Both approximation are derived as first order expansions in T, so they are expected to become less accurate for large values of T, which is indeed the case.

3 Managing smile risk

We discuss now the financial meaning of the SABR parameters and their use for smile-risk management. The common rule of thumb when talking about smile risk in Sabr model, is that α controls the level of the smile, or the ATM volatility, β and ρ the slope of the smile, and ν its convexity.

We can be more precise, and notice that the level of the smile can be associated with the initial value of the instantaneous volatility $\alpha\, S^{\beta-1}$; its slope can be associated to a skew parameter χ defined as the first derivative with respect to the logarithm of the underlying of the expected value of the instantaneous volatility given the underlying; its convexity can be associated to the vol-of-vol component γ orthogonal to the underlying. We have

$$\chi = \frac{\partial}{\partial x} \mathrm{E}[\alpha\, S^{\beta-1} | S = e^x] \, = (\beta - 1)\, \alpha\, S(0)^{\beta-1} + \rho\, \nu, \tag{10}$$

$$\gamma = \nu \sqrt{1-\rho^2}. \tag{11}$$

To manage smile risk using the sensitivities to skew (χ) and convexity (γ) instead of to correlation (ρ) and vol-of-vol (ν), would be consistent with the FX market practice of managing smile risk using sensitivities to risk-reversals and butterflies, and would be similar to managing vanna and volga.

Note that a portfolio which is perfectly hedged (to first order) with respect to α, β, ρ, ν, turns out to be also perfectly hedged with respect to α, β, χ, γ, and viceversa, since there is a C^1 bijection between the two parameterisations. However, when a portfolio is not perfectly hedged (which is often the case), measuring smile risk using sensitivities to χ and γ may give a better measure of the residual risk and more transparent hedging prescriptions. Furthermore, using sensitivities to χ and ρ may result in a better choice within Value At Risk calculations, since for example a large historical move in correlation at a time of low (implied) vol-of-vol, which had only a modest price impact at the time (note that ρ is multiplied by ν in (10)), could be projected into an arguably too large P&L move if the current vol-of-vol level is high, and viceversa.

4 Issues with the sensitivity to correlation

Within the SABR model, if ρ increases while all other parameters are kept constant we would expect the slope of the smile to increase. The same can be said if the ATM volatility is kept constant instead of the instantaneous volatility parameter α. This behaviour is indeed observed when vanilla options prices are computed using numerical techniques, and it is shared by other stochastic volatility models, e.g. Heston. In other words, we expect the sensitivity to a change in the correlation parameter ρ (rega) to be positive for high strikes, i.e. strikes on the right side of the forward, and to be negative for low strikes. The following figures display the rega (measured as the change in BS volatility resulting from a 0.01 increase in ρ) for swaptions in USD as a function of strike, for different expiries and using recent market data.

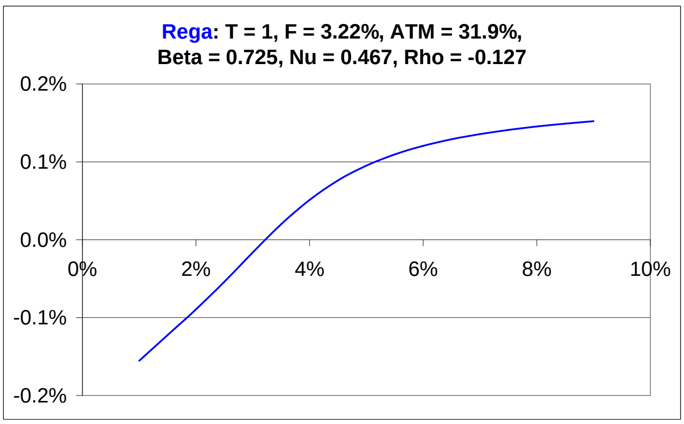

Figure 1: rega 1y-10y: for a short expiry the shape of the rega is as expected.

However, Sabr formula can behave differently for long expiries.

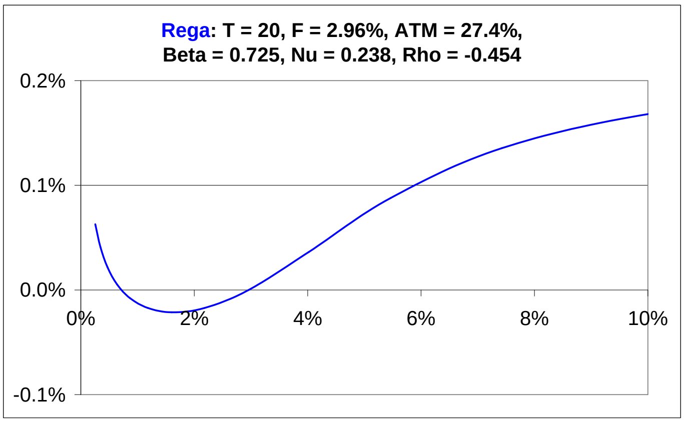

Figure 2: rega 20y-10y: for a long expiry the shape of the rega is wrong.

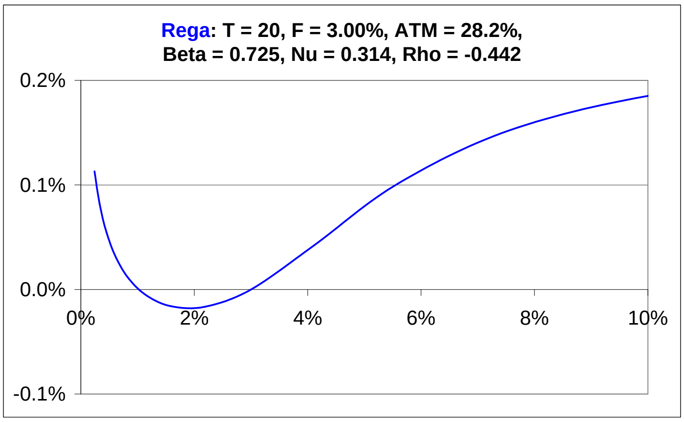

Figure 3: rega 20y-1y: for the same long expiry but a different tenor, the shape of the rega is also wrong.

Figure 4 is obtained by increasing the ATM volatility by 10%, and shows that the effects that are observed would be even more pronounced in higher volatility environments.

When managing smile risk for a single expiry, non-intuitive sensitivities to correlation (and to vol-of-vol) can make it difficult to understand the smile risk of a portfolio. Note however that if a portfolio is perfectly hedged (to first order) with respect to the SABR formula's parameters, it remains perfectly hedged (to first order) with respect to any other, possibly more meaningful, smile parameterization that would generate the same smile as the SABR formula, which mitigates the impact of the SABR formula's inaccuracy.

However, risk aggregation across different expiries can be problematic, and naively adding up correlation sensitivities over different expiries could give the wrong hedging prescriptions. To give a concrete example, using recent market data, a long position in a 20y vega weighted put spread on the 1y swap rate (long 1.1% receiver, short 0.5% receiver) would appear to be nearly perfectly hedged by a short position in a 10y call spread on 1y swap rate (short 9% payer, long 10% payer) and ATM swaptions, which is arguably wrong since both spread positions are expected to increase in value as the skew (slope

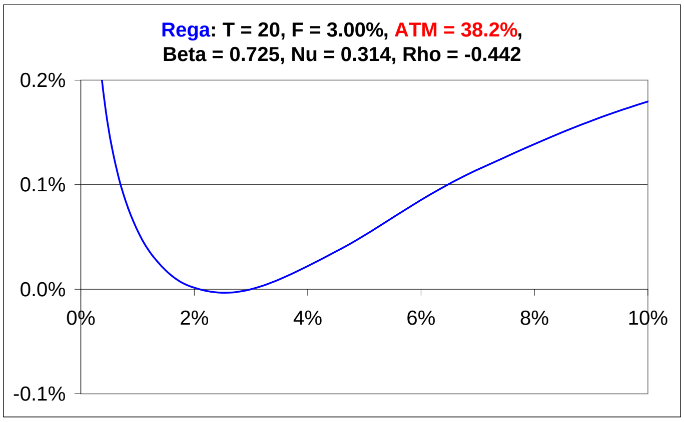

Figure 4: rega 20y-1y, ATM+10%: in a higher volatility environment the shape of the rega would be even worse.

of the smile) increases. Note that the we would face a similar problem even if were we hedging with respect to skew and convexity instead of correlation and vol-of-vol. Indeed, in the simple case where the Sabr parameters for the two expiries are the same, a portfolio would be perfectly hedged (to first order) with respect to ATM vol, skew and convexity if and only if it is perfectly hedged with respect to ATM vol, correlation and vol-of-vol (assuming β is fixed).

5 Possible solutions

Looking for a more robust risk management framework, a few solutions spring to mind.

An obvious one would be to use a different model, e.g. Heston, or a different smile parameterization, e.g. SVI. A quite comprehensive review of the different approaches to smile modelling that are available can be found for example in [2].

Remaining within the SABR model, a possible solution would be to compute vanilla option prices using a numerical technique such as that of conditional integration [3] instead of the SABR formula.

Alternatively, without departing too much (if at all) from the smile generated using the SABR formula, we could consider using a "SABR inspired" smile parameterization:

$$\Sigma(S, K, T) = I_0, \tag{12}$$

where I_0 is given by either (4) or (8). This solution can generate exactly the same smile (through different parameters) as the standard SABR model when $\beta = 1$. Note that this is achieved by using α' and ν' in (4) defined as:

$$\alpha' = \alpha(1 + I_1 T), \tag{13}$$

$$\nu' = \nu/(1 + I_1 T). \tag{14}$$

In the general case the reduced formula cannot generate a perfect match to a smile generated using the standard SABR formula, but it can get very close. The reduced formula does not seem to experience the same problems as the original SABR formula as far as rega is concerned, as figure 6 shows for the case of the 20y-1y USD swaptions. The reduced Sabr smile is calibrated to the SABR smile, and figure 5 shows the quality of the smile fit.

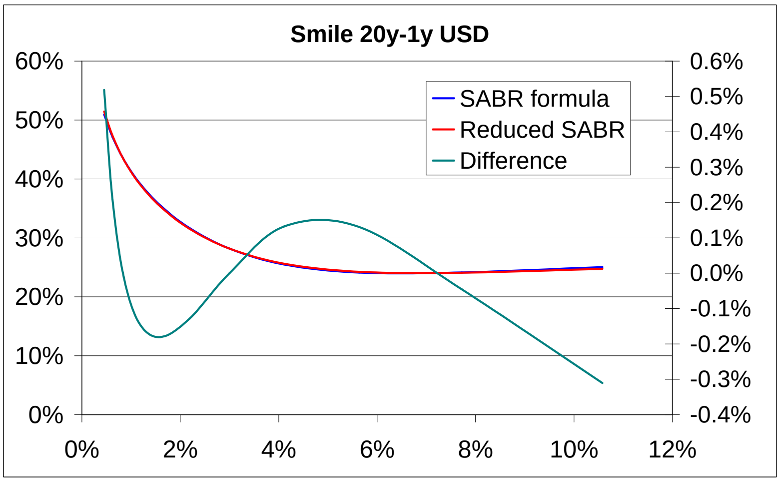

Figure 5: smile fit 20y-1y: the reduced SABR formula can be very close to the SABR formula even when $\beta \neq 1$.

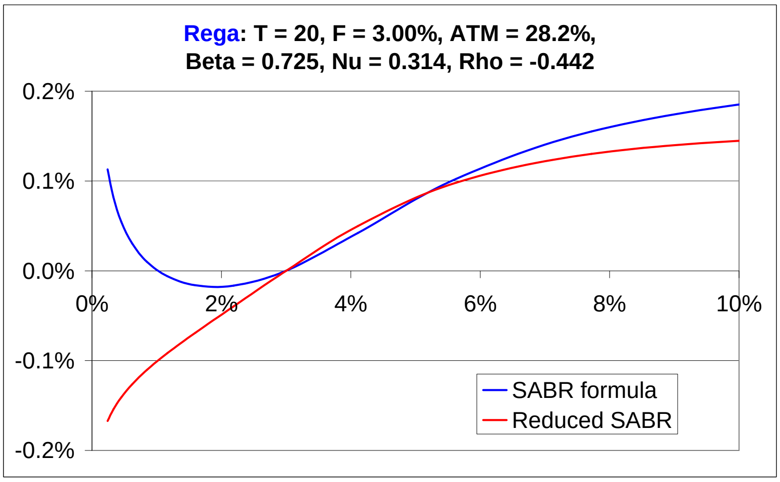

Figure 6: rega 20y-1y: the reduced SABR formula generates a sensible rega.

References

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[3] W. McGhee, *An Efficient Implemetation of Stochastic Volatility by the method of Conditional Integration*. ICBI Conference, Paris, 2011.

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