Drift Approximations in a Forward-Rate-Based LIBOR Market Model

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Abstract

In a market model of forward interest rates, a specification of the volatility structure of the forward rates uniquely determines their instantaneous drifts via the no-arbitrage condition. The resulting drifts are state-dependent and are sufficiently complicated that an explicit solution to the forward rate stochastic differential equations cannot be obtained. The lack of an analytic solution could be a major obstacle when pricing derivatives using Monte Carlo if it implied that the market could only be accurately evolved using small time steps. In this paper we use a predictor-corrector method to approximate the solutions to the forward rate SDEs and demonstrate that the market can be accurately evolved as far as twenty years in one step.

1 Introduction

The LIBOR market model, or BGM/J, approach to pricing exotic interest-rate derivatives has become very popular in recent years [BGM97, Jam97]. To price an exotic interest-rate derivative, we evaluate its risk-neutral expectation as a function of a finite set of forward rates, which move according to geometric Brownian motion with an indirectly stochastic drift, on a discrete set of times. Most interest-rate derivatives, including Bermudan swaptions and trigger swaps, can be fitted into this framework. The main difficulty in implementation is that forward-rates are not tradable assets and therefore need not be martingales in the risk-neutral measure. Indeed, in general, their drifts are not just non-zero but also state-dependent which gives rise to non-lognormal distributions in the terminal measure. This means that there is no analytic solution to the stochastic differential equation describing the forward rates' evolution. Therefore, when pricing via Monte Carlo, we must numerically approximate the probability density function. One solution is to small-step using an Euler scheme which is the commonly used approach by practitioners, albeit that this involves an obvious time-penalty. Despite the availability of various textbooks on the numerical solution of stochastic differential equations such as the classic work by Kloeden and Platen [KP99], very little has been published on improvements over the simple Euler method in the context of the BGM/J market model. An exception to this is a research paper by Kurbanmuradov et. al. [KSS99], in which the authors, nonetheless, only discuss approximations that either result in terminal distributions that are log-normal or, again, require frequent sub-stepping in order to reduce the simulation bias to a tolerable level. Here, we present a method that allows long steps and does not result in a log-normal probability density function. The approach we take is to approximate

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the drift over the mentioned long time steps, which reduces the calculation time considerably. To substantiate our claims, we demonstrate that the new method is indeed sufficiently accurate to be used in pricing. In particular, we show that the forward rates can be evolved over a 20 year time horizon in a single step.

We consider a LIBOR market model based on the N forward LIBOR rates f_i $(i = 0, .., N-1)$ that each span a time period from t_i to t_{i+1}. Let Z_i be the price of a zero-coupon bond expiring at time t_i. Assume that each forward rate f_i is driven by a standard Brownian motion W_i with a time dependent log-normal volatility and a time dependent instantaneous correlation structure $\mathrm{E}\left[dW_i dW_j\right] = \rho_{ij} dt$. The forward rate dynamics are then

$$\frac{df_i}{f_i} = \mu_i dt + \sigma_i dW_i, \tag{1}$$

where the drift μ_i is determined by the no-arbitrage conditions and will depend on the choice of probability measure. If we take the numéraire to be the zero-coupon bond expiring at one of the reset times t_η, the instantaneous drifts are

$$\mu_i\left(\boldsymbol{f}(t), t\right) = \underbrace{-\sigma_i \sum_{k=i+1}^{\eta-1} \frac{f_k(t)\tau_k}{1 + f_k(t)\tau_k} \sigma_k \rho_{ik}}_{\text{non-zero for } i<\eta-1} + \underbrace{\sigma_i \sum_{k=\eta}^{i} \frac{f_k(t)\tau_k}{1 + f_k(t)\tau_k} \sigma_k \rho_{ik}}_{\text{non-zero for } i\geq\eta} \quad . \tag{2}$$

2 Numerical Algorithm

We wish to evolve the system of forward rates from their initial values $f_i(0)$ at time 0 to some time t in the future. For simplicity of notation, all N Brownian motions are used to drive the simulation, although in practice fewer could be used. As it is simpler to work in log space, we define $Y_k = \log f_k$ and obtain the system of stochastic differential equations

$$dY_k = \left[\mu_k(Y, t) - \frac{1}{2}\sigma_k^2\right] dt + \sigma_k dW_k \ . \tag{3}$$

A single Euler step to evolve all of the state variates Y_k from time 0 to time t is given by

$$Y_k^E(t) = Y_k(0) + \left[\mu_k\left(Y(0), 0\right) - \frac{1}{2}\sigma_k^2(0)\right] t + \sigma_k(0) \cdot \sqrt{t} \cdot \hat{z}_k \tag{4}$$

where $\hat{z}_k$ are $N(0, 1)$ random variables which are correlated according to $\mathrm{E}\left[z_i z_j\right] = \rho_{ij}(0)$. For time dependent instantaneous volatility, the simple Euler method (4) is improved by the use of the integrated covariance matrix elements $C_{ij} = \int_0^t \rho_{ij}(s)\sigma_i(s)\sigma_j(s)ds$ which can be split into its pseudo-square root A defined by $C_{ik} = \sum_{j=0}^{N-1} A_{ij} A_{kj}$. There are a variety of methods that can be used to generate a valid matrix A_{ij} such as Cholesky, spectral, or angular decomposition [PTVF92, RJ00]. Using this definition of the integrated covariance matrix C, we can express an improved constant drift approximation $\hat{\mu}_k(Y, C)$ as

$$\hat{\mu}_k(Y, C) = -\sum_{j=k+1}^{\eta-1} C_{ij} \frac{\tau_j e^{Y_j}}{1 + \tau_j e^{Y_j}} + \sum_{j=\eta}^{i} C_{ij} \frac{\tau_j e^{Y_j}}{1 + \tau_j e^{Y_j}} \quad . \tag{5}$$

This leads to what we refer to as the *log-Euler* scheme,

$$Y_k^E(t) = Y_k(0) + [\hat{\mu}_k(Y(0), C) - \frac{1}{2}C_{kk}] + \sum_{j=0}^{N-1} A_{kj} z_j \ , \tag{6}$$

with all of the z_j now being independent $N(0,1)$ random variates. This algorithm does not attempt to account for the state dependence of the drifts, but instead uses the initial values of the (logarithms of the) forward rates in the formulas (5). However, the results produced by the log-Euler method will be useful to compare against our improved method. Note that the dimension of t is in equation (6) accounted for by the fact that C is an integral over time, and A is by dimension a square root of C and thus of dimension $\sqrt{t}$.

There are a number of ways to improve on the Euler method for the numerical integration of stochastic differential equations, many of which may be found in the canonical reference by Kloeden and Platen [KP99]. Instead of using any of the well-known explicit, implicit, or standard predictor-corrector methods, we employ a hybrid technique whereby we integrate the terms $\sigma_k dW_k$ directly as if the drift coefficient is constant over any one time step. This is essentially consistent with the standard Euler method. However, in addition, we account for the indirect stochasticity of the drift term by using a *Predictor-Corrector* method. First, we predict the terminal values of the forward rates using the initial data, and then use these values to correct the approximation in the drift coefficient. Our algorithm for constructing one draw from the terminal distribution of the forward rates over one time step is thus as follows.

1. Evolve the logarithms of the forward rates as if the drifts were constant and equal to their initial values according to the log-Euler scheme (6).
2. Compute the drifts at the terminal time with the so evolved forward rates.
3. Average the initially calculated drift coefficients with the newly computed ones.
4. Re-evolve using the same normal variates as initially but using the new predictor-corrector drift terms.

This is a very natural and simple way to incorporate the drift state-dependence. In the special case, rarely arising in finance, that the volatilities are constant, this is the simplest predictor-corrector method (equation (15.5.4) with $\alpha = 1/2$ in [KP99])

$$Y_k(t) \quad = \quad Y_k(0) + \frac{1}{2}[\hat{\mu}_k(Y^E(t), C) + \hat{\mu}_k(Y(0), C) - C_{kk}] + \sum_{j=0}^{N-1} A_{kj} z_j, \qquad (7)$$

whilst in the general case, we have a new hybrid method. Note that this hybrid depends heavily on the fact that the SDE for the constant drift case is solvable. While further, potentially better, approximations are possible, our approach has the advantage of being very simple to understand and implement, and as we shall show in the next section, it is able to reproduce with a high degree of accuracy both the forward rate probability distributions and derivative prices for actual market data.

3 The explicit LIBOR-in-arrears density

When testing our numercial algorithm, it will be useful to have a non-trivial test case which can be solved analytically for comparison. In order to avoid having to call on theoretical results, we give a simple deduction of the risk-neutral density using the well-known fact that the risk-neutral density is the second derivative of the call option price with respect to the strike.

Define $\Phi_{i,\eta}$ to be the terminal probability distribution of the forward rate f_i in the measure where Z_η is the numéraire. Choosing $\eta := i+1$ as the numéraire index yields the price of a caplet on f_i struck at K as

$$C(0) = Z_{i+1}(0) \int_0^\infty \tau_i (f_i - K)^+ \Phi_{i,i+1}(f_i)\, df_i \ . \tag{8}$$

In this measure, f_i is a martingale and has a log-normal probability distribution

$$\Phi_{i,i+1}(f) = \frac{1}{\sqrt{2\pi}} \frac{e^{-\frac{1}{2}\frac{\left[\ln\left(\frac{f}{f_i(0)}\right)+\frac{1}{2}\bar{\sigma}_i^2 t_i\right]^2}{\bar{\sigma}_i^2 t_i}}}{f\,\bar{\sigma}_i\sqrt{t_i}} \qquad \text{with} \qquad \bar{\sigma}_i^2 t_i = \int_0^{t_i} \sigma_i^2(s)\, ds. \tag{9}$$

If we instead use Z_i as the numéraire, the caplet price is

$$C(0) = Z_i(0) \int_0^\infty \tau_i (f - K)^+ \frac{\Phi_{i,i}(f)}{1+\tau_i f}\, df \ . \tag{10}$$

The density $\Phi_{i,i}$ is also known as the *LIBOR-in-arrears* density due to it being precisely the distribution required to value LIBOR-in-arrears contracts which pay immediately upon the setting of a FRA, not at maturity of the FRA itself. Equating the two call option prices (8) and (10) and differentiating them twice with respect to the strike yields

$$\Phi_{i,i}(f) = \frac{1+\tau_i f}{1+\tau_i f_i(0)} \Phi_{i,i+1}(f) \ . \tag{11}$$

Thus, we have an analytic value for the terminal probability density of f_η (recall that η is the index of the payment time of the numéraire). We can explicitly calculate the moments of this distribution. It follows from equation (11) that if $\mathrm{E}^i[\cdot]$ is expectation in the measure where Z_i is the numéraire, then

$$\mathrm{E}^i[g(f_i)] = \frac{\mathrm{E}^{i+1}[g(f_i)] + \tau_i \mathrm{E}^{i+1}[f_i g(f_i)]}{1+\tau_i f_i(0)}. \tag{12}$$

Since the moments of the log-normal forward rate f_i in its natural numéraire Z_{i+1} are given by $\mathrm{E}^{i+1}[f_i^n] = f_i(0)^n e^{\frac{1}{2}n(n-1)\bar{\sigma}_i^2 t_i}$, we have for the LIBOR-in-arrears case

$$\mathrm{E}^i[f_i^n] = f_i(0)^n e^{\frac{1}{2}n(n-1)\bar{\sigma}_i^2 t_i} \frac{1+\tau_i f_i(0) e^{n\bar{\sigma}_i^2 t_i}}{1+\tau_i f_i(0)} \tag{13}$$

which we will use as an analytical test for the accuracy of our numerical approximations.

4 Numerical tests for the Libor-in-arrears case

The covariance structure we use was suggested by Rebonato [Reb98]. The instantaneous forward-rate volatilities are

$$\sigma_i(s) = \begin{cases} k_i \left\{ [a + b(t_i - s)]\, e^{-c(t_i - s)} + d \right\} & \text{for} \quad s < t_i \\ 0 & \text{for} \quad s > t_i \end{cases} , \tag{14}$$

where a, b, c and d are obtained from a least-squares fit to the market caplet volatilities, and the k_i's are fixed by requiring that the caplet prices be recovered exactly. The instantaneous correlation between forward rates is given by the time-independent function $\rho_{ij} = e^{-\beta|t_i - t_j|}$ with $\beta = 0.1$. In figure 1, we show the probability densities for the exact, Euler, and predictor-corrector methods for the LIBOR-in-arrears case. As can be seen, the density resulting from the predictor-corrector

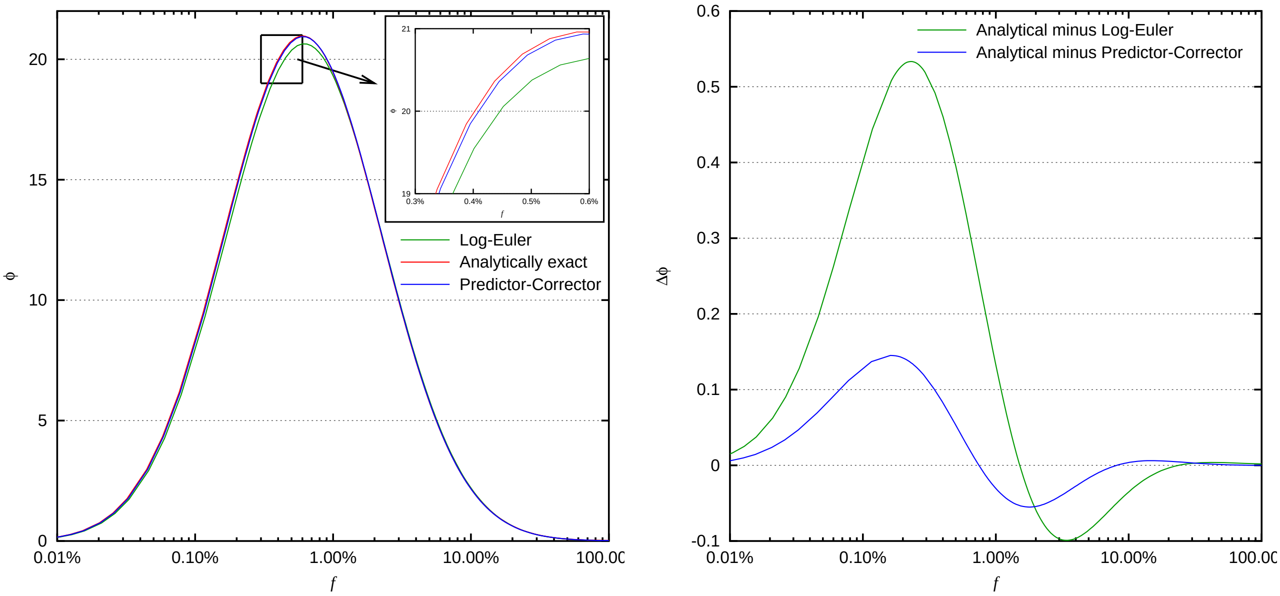

Figure 1: Plot of probability densities for f_η (LIBOR-in-arrears numéraire case) and the differences from the exact solution. The 3-month forward rate value was 8%, the volatility was set at constant 24% for an expiry of 30 years.

method is virtually identical to the analytical solution in a direct comparison. A closer look at the actual differences reveals that the initial value (Euler) method incurs not only a much larger error on an absolute scale, it also deviates from the analytically exact solution on a wider interval. In other words, the error from the predictor-corrector method is more localised on the distribution's abscissa which yields to a much better match of the moments of the distribution.

Method	$\frac{\mathrm{E}[f]}{f_0}$	$\frac{\mathrm{E}[f^2]}{f_0^2}$	$\frac{\mathrm{E}[f^3]}{f_0^3}$	$\frac{\mathrm{E}[f^4]}{f_0^4}$
Scenario	$f_0 = 8\%, \sigma = 24\%, T = 30$			
Exact	1.09077	9.01694	798.912	657870.
Euler	1.03446	6.02408	197.482	36443.9
Predictor-Corrector	1.09896	10.0732	1020.33	699711.
Scenario	$f_0 = 8\%, \sigma = 14\%, T = 20$			
Exact	1.00941	1.51448	3.38384	11.2888
Euler	1.00772	1.50286	3.31699	10.8346
Predictor-Corrector	1.00950	1.51590	3.39706	11.4146
Scenario	$f_0 = 8\%, \sigma = 14\%, T = 10$			
Exact	1.00425	1.22798	1.82864	3.31703
Euler	1.00385	1.22591	1.82126	3.29160
Predictor-Corrector	1.00426	1.22810	1.82935	3.32039

Table 1: Table of moments for three different LIBOR-in-arrears numéraire calculations.

As was discussed in section 3, we have analytic formulas for the moments of the distribution. For a constant drift μ_η, the moments resulting from the single-step Euler approximation are

$$E^i[(f_i^E)^n] = f_i(0)^n e^{n\mu(0)} e^{\frac{1}{2}n(n-1)\bar{\sigma}_i^2 t_i} \ . \tag{15}$$

However, an analytic expression for the predictor-correc-tor method is not possible, and the moments must be calculated numerically[1]. In table 1 we summarize the moments of the distributions for a small set of parameter scenarios. To conclude the tests for the LIBOR-in-arrears case, we present two more diagrams in figure 2. The graphs show by how much the cash flow associated with a single forward rate agreement is mispriced in the Libor in arrears case when a single step is taken out to its reset time. Since we are interested in the error in the forward rate agreement itself, the discounting that was applied to the cashflow in the expectation calculation was undone, i.e. the figures show $\mathrm{E}\left[\frac{f_\eta(T)Z_\eta(T)}{Z_\eta(0)}\right] - f_\eta$. In order to give the reader a feeling for the significance of the deviation, the (non-discounted) vega curve of the at-the-money caplet was added to the diagrams. Figure 2 makes the superior performance of the

[1] We used the `NIntegrate` function of Mathematica 3.0 for this purpose.

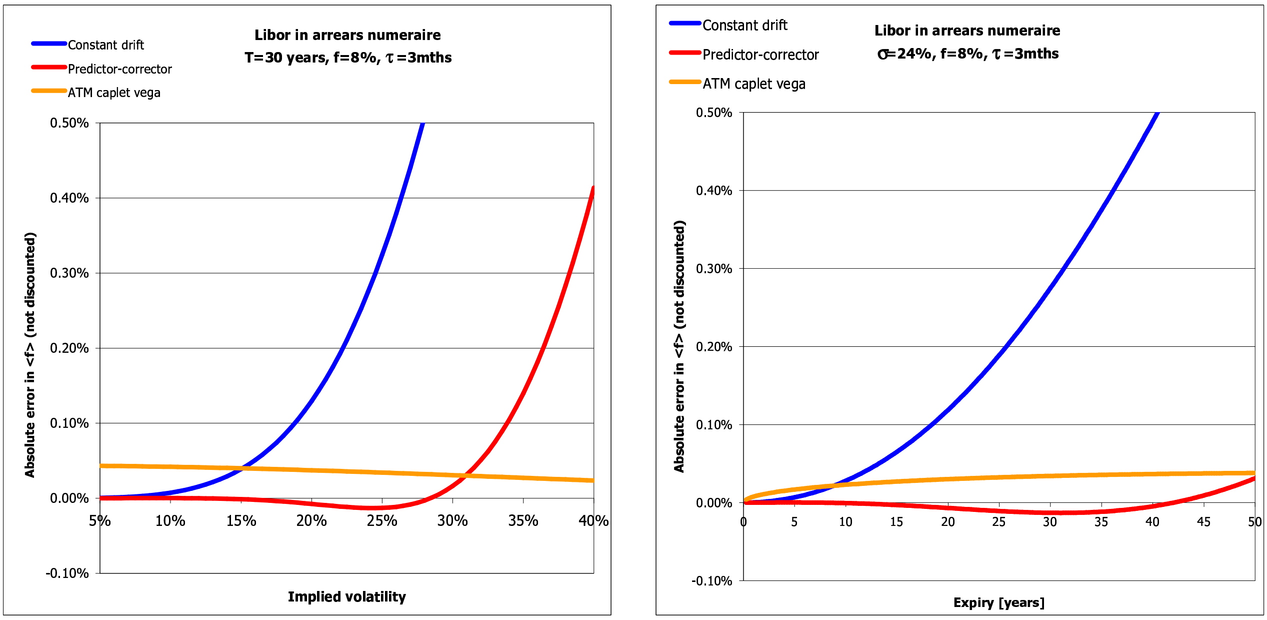

Figure 2: The error in the (un-discounted) FRA for the LIBOR-in-arrears numéraire as a function of time to expiry and volatility.

predictor-corrector method over the initial-value-constant-drift approach very clear: for volatilities as high as 24% one could take steps of up to 45 years and incur a maximum error of a couple of basis points in the forward rate which constitutes the typical bid-offer spread of a FRA. Of course, with increasing volatility and time to expiry, the approximation will deteriorate. It is striking, however, how much the drift error of a single-step simulation can be reduced by the predictor-corrector drift method.

5 Numerical tests for far-out numéraires

In figure 3, we present the error incurred by a single-step Monte Carlo simulation for a whole sequence of individual 3-month forward-rate agreements. The data used for this calculation were the rates and volatilities as they prevailed in the Sterling market on the 23$^{\text{rd}}$ of October 2000. The calibration resulted in the values $a = -6.29\%$, $b = 11.7\%$, $c = 0.6$, and $d = 14.8\%$ to be used in equation (14), and a vector of 79 values for the k_i (all being near 1) which we can supply on demand[2]. Again, we give the (non-discounted) vega curve of at-the-money caplets for comparison. All of the calculations were done with a zero coupon bond of 19.75 years to maturity as numéraire. For each calculation, 131072 sample paths were used in conjunction with a Sobol' sequence generator, and all of the simulations were well converged. Any slightly non-smooth variations of the curves are a consequence of the variations of both the actual yield curve and the caplet implied volatilities on that day. One can clearly see in figure 3 that the drift error of the single step Euler method for short-dated caplets is negligible. In other words, for the given setting, the drift error exceeds 2 basis points only when an initial step size of around 3 years is exceeded. Beyond that, the Euler-stepping drift error becomes quite marked, peaking at around 12 basis points for 12-year steps. This is to be compared with an absolute error of less than 2 basis points of the predictor-corrector method even for such large steps in time. As the expiry of the caplet approaches the maturity of the numéraire bond, the drift errors decrease again, to be zero exactly when the expiry

[2]The market data used for the calculation can be found in appendix A.

time plus accrual period of the caplet equals the maturity of the numéraire bond (in which case we speak of the *natural numéraire of the FRA*). The very last data point in the diagram is a caplet with expiry in 19.75 years, and payment in 20 years, which is again the familiar LIBOR-in-arrears numéraire case.

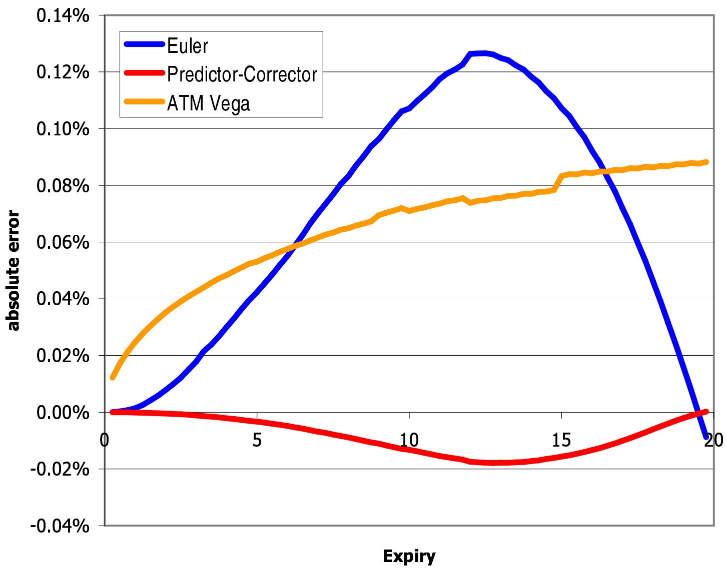

Figure 3: FRA errors for one and the same long bond (19.75 years) as numéraire from a single step Monte Carlo simulation using either the (log-)Euler scheme or the predictor-corrector scheme.

Trigger swaps, amongst other derivatives, lend themselves particularly well to simulation out to a single time horizon, that is a single step Monte Carlo simulation. This is due to the fact that the choice of a long dated zero coupon bond (maturing on or after the reset time of the last trigger FRA) as numéraire enables us to forward all cash flows caused by a trigger event to maturity of the numéraire bond using the reset (i.e. fixing) value of each of the forward rates. By virtue of the high accuracy of the predictor-corrector drift approximation, we do not need to step through all the time horizons of the reset times of all the forward rates. Instead, we simply let the instantaneous volatility curve of each forward rate drop to zero at the time of its fixing. Using this approach, we can calculate the covariance matrix of the evolution of the forward rates from the time of inception of the contract directly out to the reset time of the last involved forward rate. In doing so we can, of course, still sub-step in order to obtain a better numerical integration of the stochastic differential equation, and we can choose the sub-stepping intervals arbitrarily rather than being forced to loop over all of the individual reset times.

In figure 4, the convergence diagrams of pricing calculations for a 20-year quarterly trigger swap is shown. This contract depends on a total of 80 underlying LIBOR rates, the first one of which had already reset, i.e. was fixed. All of the remaining 79 forward rates were allowed their own individual (correlated) Wiener process, which means that the trigger swap was priced with a 79 factor model. The market parameters and the yield curve are the same as those used for the previous calculations whose results were shown in figure 3. The trigger swap starts with the first trigger opportunity being in 3 months from now, and then every three months until the last possibility being in 19.75 years from now. The strike was set at 6.5% and the trigger level (above which any single resetting 3-month LIBOR rate has to be in order to trigger in the swap) was at 7.5%.

Each curve in the diagrams represents the convergence behaviour for a given number of sub-steps out to the time horizon in 19.75 years from inception. Note that the scale of the abscissa of the convergence diagram is logarithmic. Since we used the Sobol' low discrepancy sequence with a Brownian bridge Wiener path construction, the conventional standard error estimate is at best a conservative figure, and at worst meaningless. Therefore, we present convergence diagrams that extend on a logarithmic scale way well beyond the point of the flattening of the convergence curve in order to provide substantial comfort that the calculation has actually converged. We also carried out simulations using other number generators and confirmed that the convergence levels

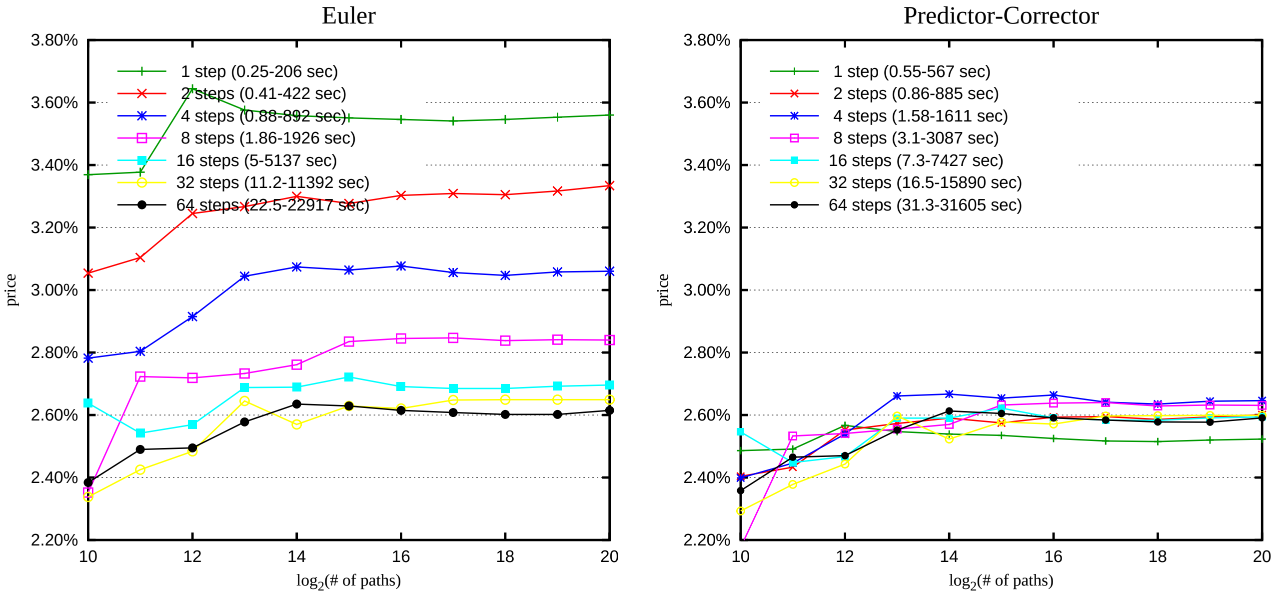

Figure 4: Convergence diagrams of a trigger swap over the same set of spanning forward rates and the same market parameters as used for figure 3. The times mentioned next to the number of sub-steps for each curve is the cpu time taken for the first (1024 sample paths) and the last (1048576 sample paths) data point on a PII@333MHz. Note that the abscissa is on a logarithmic scale. Whilst this means that many of the calculations were carried out to excessive accuracy and thus required much longer cpu times than acceptable in a trading environment, it provides substantial confidence that the curves are well converged.

are the same, albeit that conventional Monte Carlo sampling took significantly longer until the error margin had sufficiently decreased. Next to the legend entry of any curve we give the cpu times the first and the last data point took to calculate on a Pentium II at 333MHz. The calculation time is almost exactly linear in both the number of steps and the number of sample paths used, with the predictor-corrector calculations taking approximately twice as long as the Euler method.

It can be seen in the diagrams in figure 4 that as little as two steps to maturity suffice for the price to converge to within 5 basis points accuracy when the predictor-corrector scheme is used, whilst 32 substeps are needed for the Euler scheme to achieve the same. Taking into account the additional effort involved in the predictor-corrector scheme, this still provides a speedup of a factor around 8. What's more, in practice, we are usually satisfied with the accuracy given by the single-step predictor-corrector scheme (which is around 8 basis points here), but we would definitely discard the single-step Euler approximation which produces an error of almost 100 basis points in the given example. Using the single step predictor-corrector method, even this extreme case of a trigger swap priced using 79 driving Brownian motions takes no longer than a few seconds on what can already be considered out-of-date computing hardware. In comparison, recombining multi-factor tree, lattice, or convolution methods, are rarely any faster than this, even though they don't allow for the fully flexible volatility structure accessible by the BGM/J modelling approach.

We also note in figure 4 that even with the predictor-corrector method there is a small but noticeable price dependence on the number of steps. This reflects the fact that we are doing a simple approximation to a subtle density which may affect the price for highly path-dependent options. Similar effects can be observed when comparing the prices of barrier options from log-normal models with the BGM/J approach where the (indirect) stochasticity of the drift terms gives rise to slight variations.

6 Conclusion

In this article, we compared the conventional log-Euler method for the integration of the stochastic differential equation arising in the BGM/J interest rate modelling framework with a *hybrid predictor-corrector drift method*. We paid particular attention to the special case of LIBOR-in-arrears, not just because it is analytically tractable and enables us to explicitly compare the probability densities, but also because it is the basic building block of all convexity corrections. Careful numerical tests in addition to our analysis provided overwhelming evidence that the new method outperforms the conventional Euler stepping approach. Our overall summary of this article is that the new predictor-corrector approach is a powerful enhancement of BGM/J Monte Carlo simulations.

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A Market data used in section 5

t	Fra	GBP - 23 / Oct / 2000 discountfactor	Number of FRAs in residual curve	Caplet implied Volatility	k
0	6.149%	1.00000			
0.25	6.121%	0.98486	79	11.17%	1.08442
0.5	6.155%	0.97002	78	11.17%	0.94146
0.75	6.178%	0.95532	77	11.14%	0.84566
1	6.214%	0.94079	76	12.40%	0.87004
1.25	6.249%	0.92640	75	14.66%	0.96865
1.5	6.265%	0.91214	74	16.34%	1.03112
1.75	6.281%	0.89808	73	17.04%	1.03724
2	6.289%	0.88420	72	17.37%	1.02821
2.25	6.279%	0.87051	71	17.48%	1.01286
2.5	6.263%	0.85705	70	17.70%	1.00898
2.75	6.247%	0.84384	69	18.16%	1.02249
3	6.227%	0.83087	68	18.37%	1.02450
3.25	6.211%	0.81813	67	19.03%	1.05468
3.5	6.215%	0.80562	66	18.57%	1.02446
3.75	6.215%	0.79330	65	18.49%	1.01718
4	6.167%	0.78116	64	18.61%	1.02229
4.25	6.166%	0.76930	63	18.51%	1.01656
4.5	6.155%	0.75762	62	18.45%	1.01350
4.75	6.155%	0.74614	61	18.30%	1.00689
5	6.081%	0.73483	60	18.16%	1.00063
5.25	6.080%	0.72383	59	17.98%	0.99318
5.5	6.056%	0.71299	58	17.84%	0.98795
5.75	6.056%	0.70236	57	17.68%	0.98215
6	6.032%	0.69188	56	17.57%	0.97903
6.25	6.032%	0.68160	55	17.53%	0.98008
6.5	6.008%	0.67148	54	17.54%	0.98375
6.75	6.008%	0.66154	53	17.57%	0.98886
7	6.000%	0.65175	52	17.49%	0.98767
7.25	6.000%	0.64212	51	17.38%	0.98489
7.5	5.975%	0.63263	50	17.34%	0.98571
7.75	5.975%	0.62332	49	17.29%	0.98660
8	5.932%	0.61415	48	17.19%	0.98416
8.25	5.932%	0.60517	47	17.17%	0.98630
8.5	5.906%	0.59633	46	17.16%	0.98882
8.75	5.906%	0.58765	45	17.14%	0.99084
9	6.004%	0.57910	44	16.77%	0.97239
9.25	6.004%	0.57054	43	16.75%	0.97412
9.5	5.991%	0.56210	42	16.74%	0.97635
9.75	5.991%	0.55381	41	16.71%	0.97757
10	5.838%	0.54563	40	16.76%	0.98302
10.25	5.835%	0.53779	39	16.71%	0.98269
10.5	5.805%	0.53005	38	16.69%	0.98442
10.75	5.805%	0.52247	37	16.65%	0.98481
11	5.793%	0.51500	36	16.70%	0.99004
11.25	5.793%	0.50765	35	16.67%	0.99028
11.5	5.765%	0.50040	34	16.64%	0.99097
11.75	5.765%	0.49329	33	16.61%	0.99170
12	5.598%	0.48628	32	17.15%	1.02578
12.25	5.598%	0.47957	31	17.06%	1.02284
12.5	5.561%	0.47295	30	17.07%	1.02569
12.75	5.561%	0.46646	29	16.99%	1.02257
13	5.518%	0.46007	28	16.93%	1.02117
13.25	5.518%	0.45381	27	16.87%	1.01938
13.5	5.480%	0.44763	26	16.81%	1.01770
13.75	5.480%	0.44158	25	16.74%	1.01557
14	5.435%	0.43561	24	16.68%	1.01325
14.25	5.435%	0.42977	23	16.60%	1.01044
14.5	5.396%	0.42401	22	16.53%	1.00770
14.75	5.396%	0.41837	21	16.47%	1.00571
15	5.390%	0.41280	20	16.35%	1.00000
15.25	5.390%	0.40731	19	16.33%	1.00000
15.5	5.339%	0.40190	18	16.31%	1.00000
15.75	5.338%	0.39660	17	16.28%	1.00000
16	5.280%	0.39138	16	16.26%	1.00000
16.25	5.280%	0.38628	15	16.24%	1.00000
16.5	5.236%	0.38125	14	16.22%	1.00000
16.75	5.236%	0.37632	13	16.20%	1.00000
17	5.191%	0.37146	12	16.18%	1.00000
17.25	5.191%	0.36670	11	16.16%	1.00000
17.5	5.150%	0.36200	10	16.14%	1.00000
17.75	5.150%	0.35740	9	16.12%	1.00000
18	5.100%	0.35286	8	16.10%	1.00000
18.25	5.100%	0.34842	7	16.09%	1.00000
18.5	5.057%	0.34403	6	16.07%	1.00000
18.75	5.057%	0.33974	5	16.05%	1.00000
19	5.024%	0.33549	4	16.04%	1.00000
19.25	5.024%	0.33133	3	16.02%	1.00000
19.5	4.976%	0.32722	2	16.00%	1.00000
19.75	4.976%	0.32320	1	15.99%	1.00000
20		0.31923			