No-Arbitrage Conditions for the Dynamics of Smiles

Presentation at King's College

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QUARC

Royal Bank of Scotland Group

Research in collaboration with Mark Joshi

Thanks to David Samuel

The Context

- Process-based pricing
- Evolution of Smile Surfaces
- Schoenbucher’s approach
- The ‘thermodynamics’ of option pricing
- The concept of model-independent arbitrage
- Links with Merton’s theory of rational option pricing

Relevance and Possible Applications

- Understanding of essential features of models producing non-flat smile surfaces
- Defining in a precise fashion concepts such as ‘floating smile’, ‘forward-propagated smile’, etc, and identifying the class of processes that can produce such smiles
- Trying to apply these results to
  - Static replication
  - Pricing of ‘forward-starting’ options

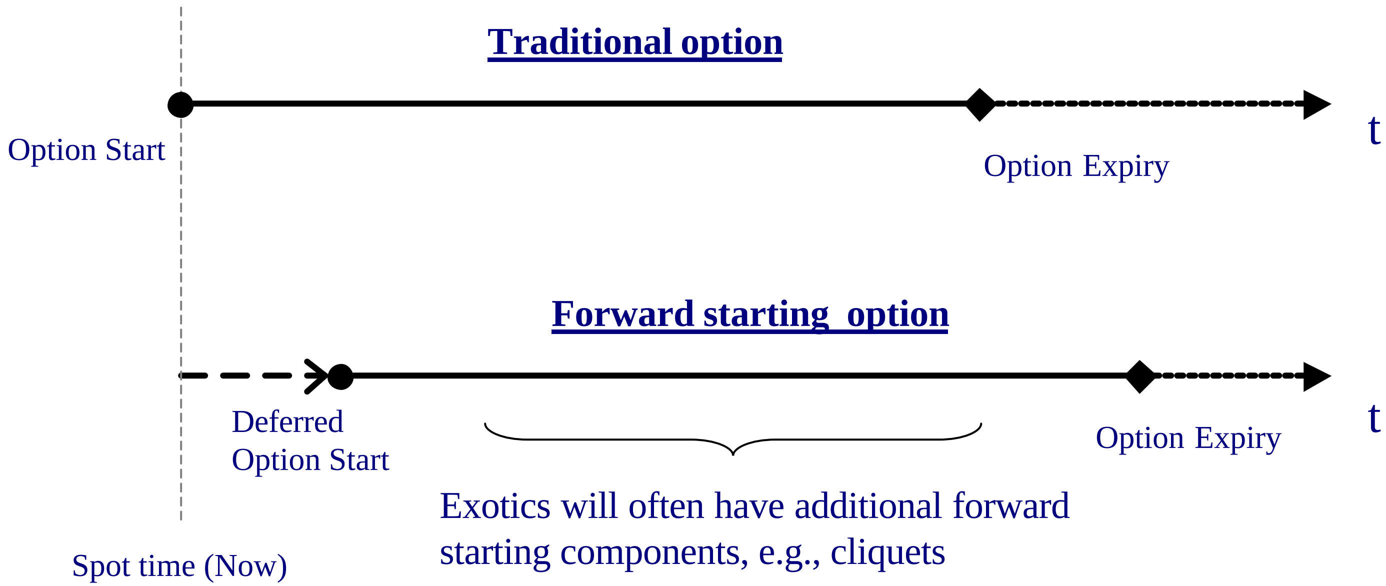
The Temptation

- How traders actually use process-based models, and how they assess their quality
- The problem of ‘forward-starting’ options
- A practitioner’s solution

Pricing Forward Starting Options (from D Samuel)

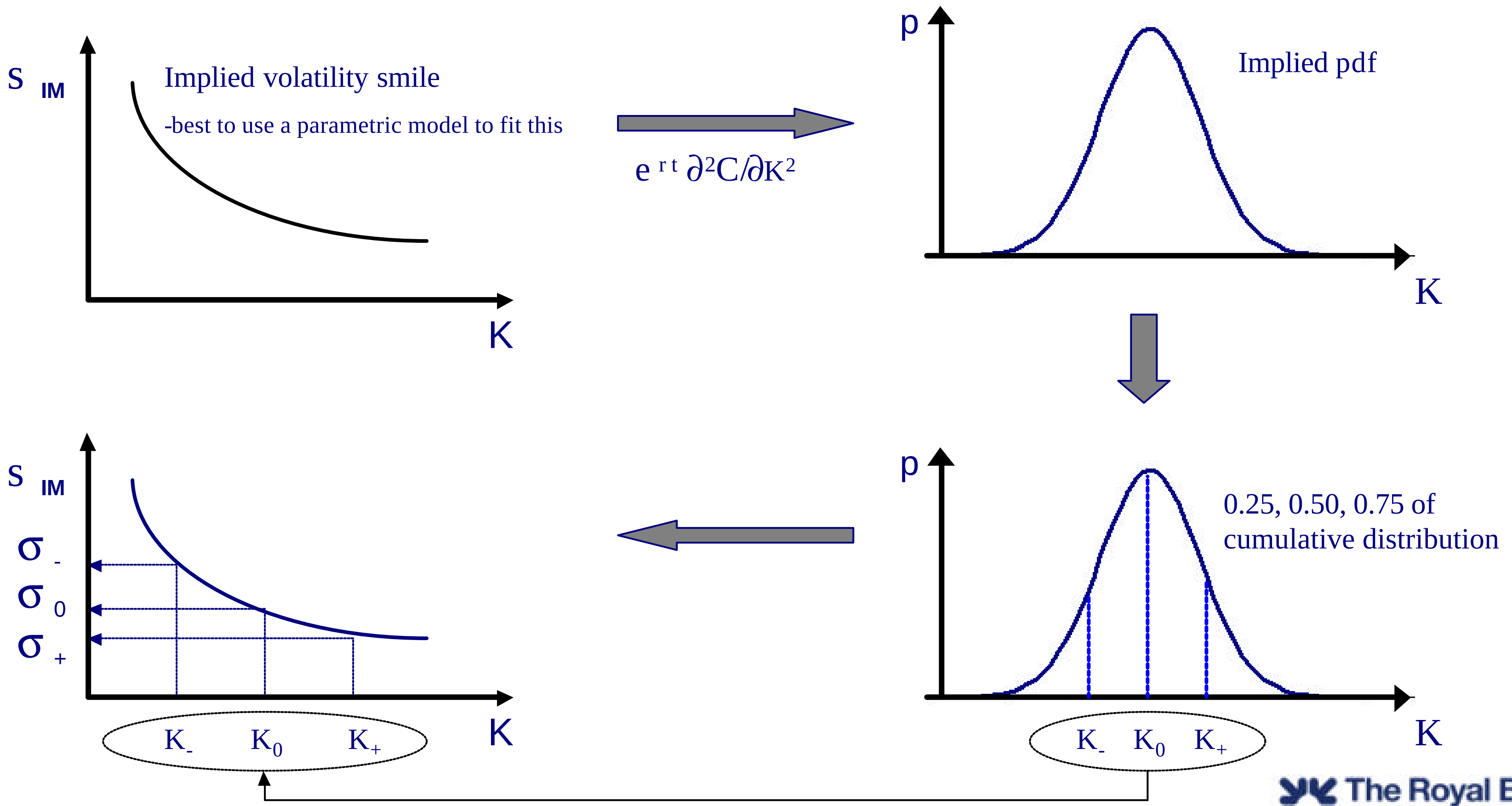
STATEMENT OF THE PROBLEM:

Products sold into the equity-linked investment market generally have a number of forward starting option features, these can be difficult to model and risk manage.



Pricing Forward Starting Options (from D Samuel)

Can we establish characteristic ‘finger-prints’ for the shape of the implied volatility surface and use these to analyse model generated forward volatility surfaces?



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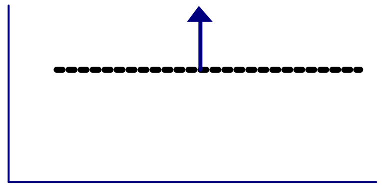
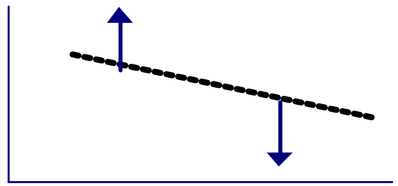
Pricing Forward Starting Options (from D Samuel)

We will then characterise the implied volatility smile at a given tenor by:

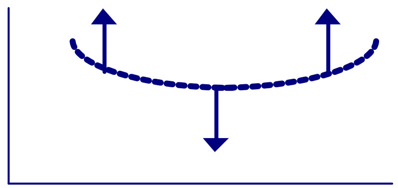
A volatility measure: σ_0



A "skew" measure: $\chi = (\sigma_+ - \sigma_-) / \sigma_0$

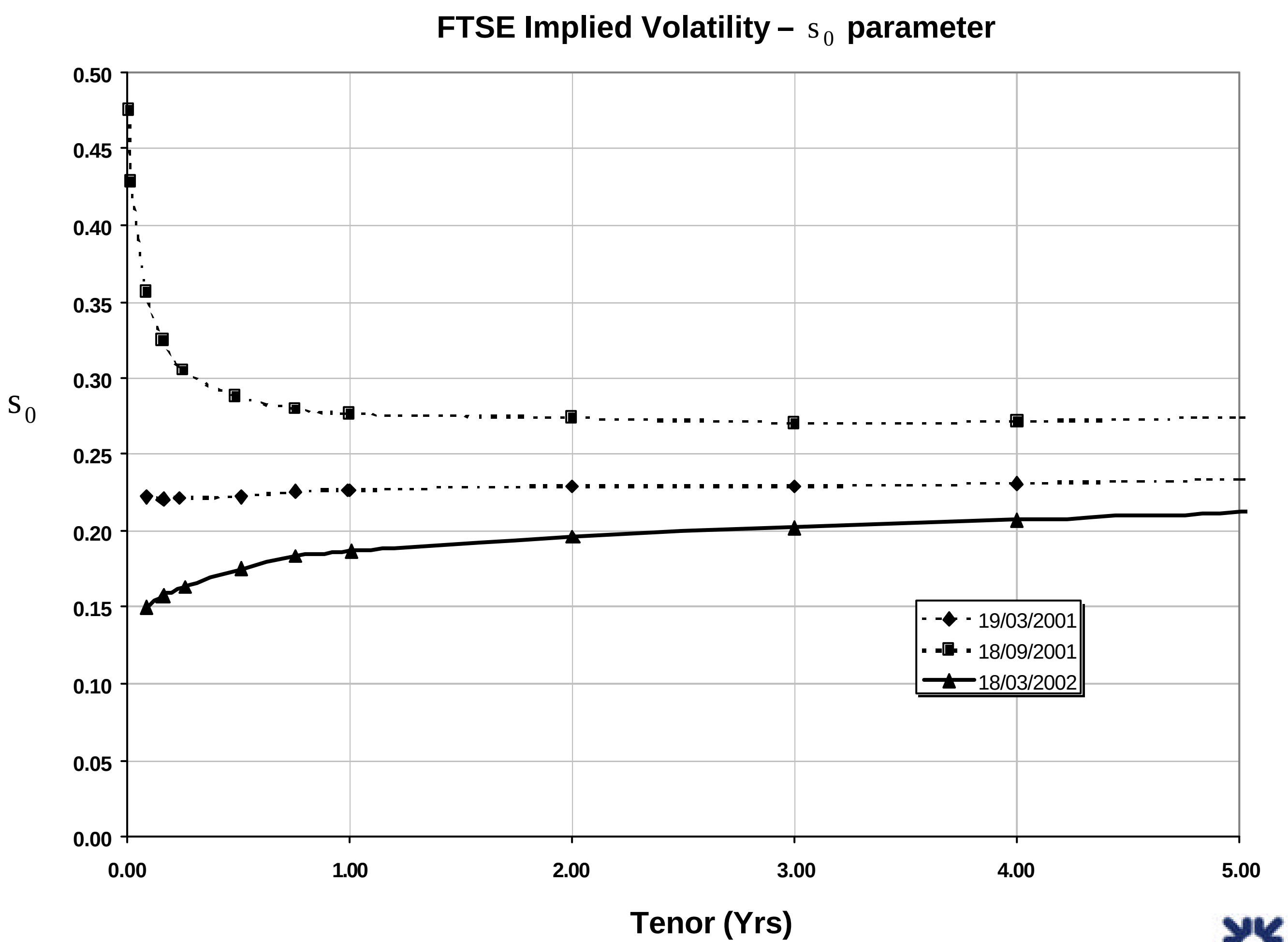


A "convexity" measure: $\omega = (\sigma_+ + \sigma_- - 2\sigma_0) / \sigma_0$



It is helpful to look at these quantities for a couple of real markets (FTSE, ESX). Graphs are shown of the above three measures for a range of tenors (1mth through to 5yr), and for three base dates: 19-Mar-01, 18-Sep-01, 18-Mar-02. The mid-date will illustrate the effect of a shock function on the equity markets.

Pricing Forward Starting Options (from D Samuel)



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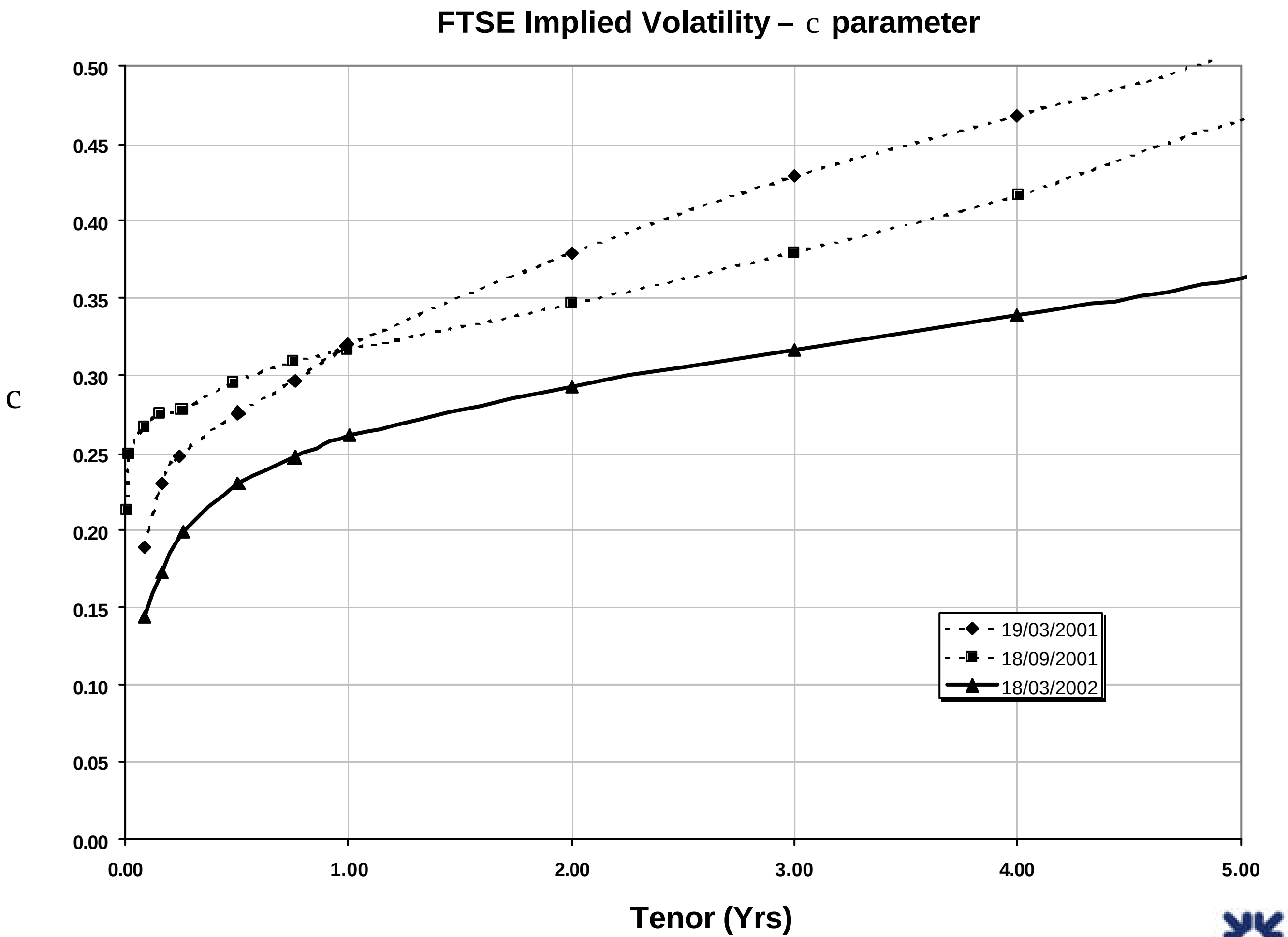

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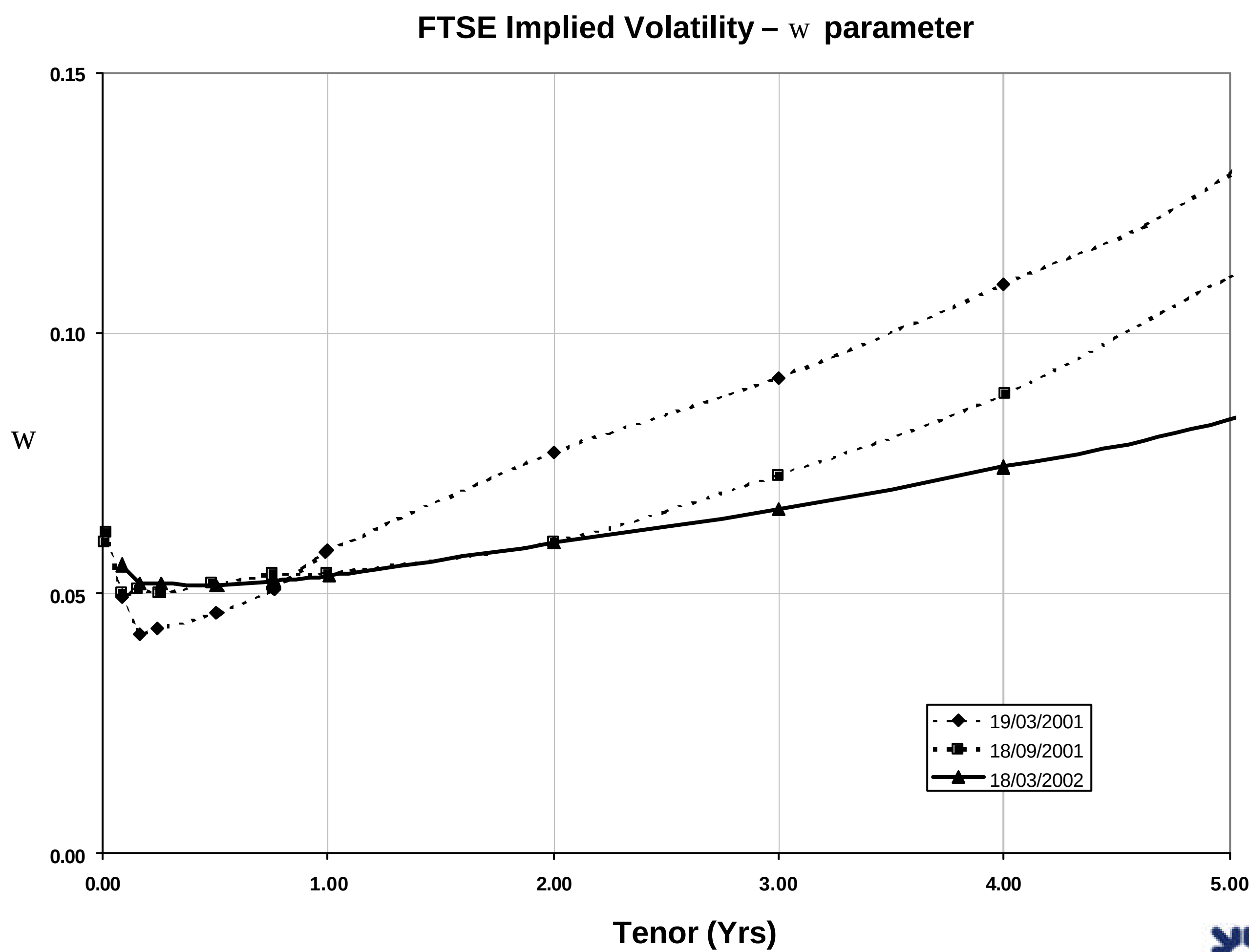

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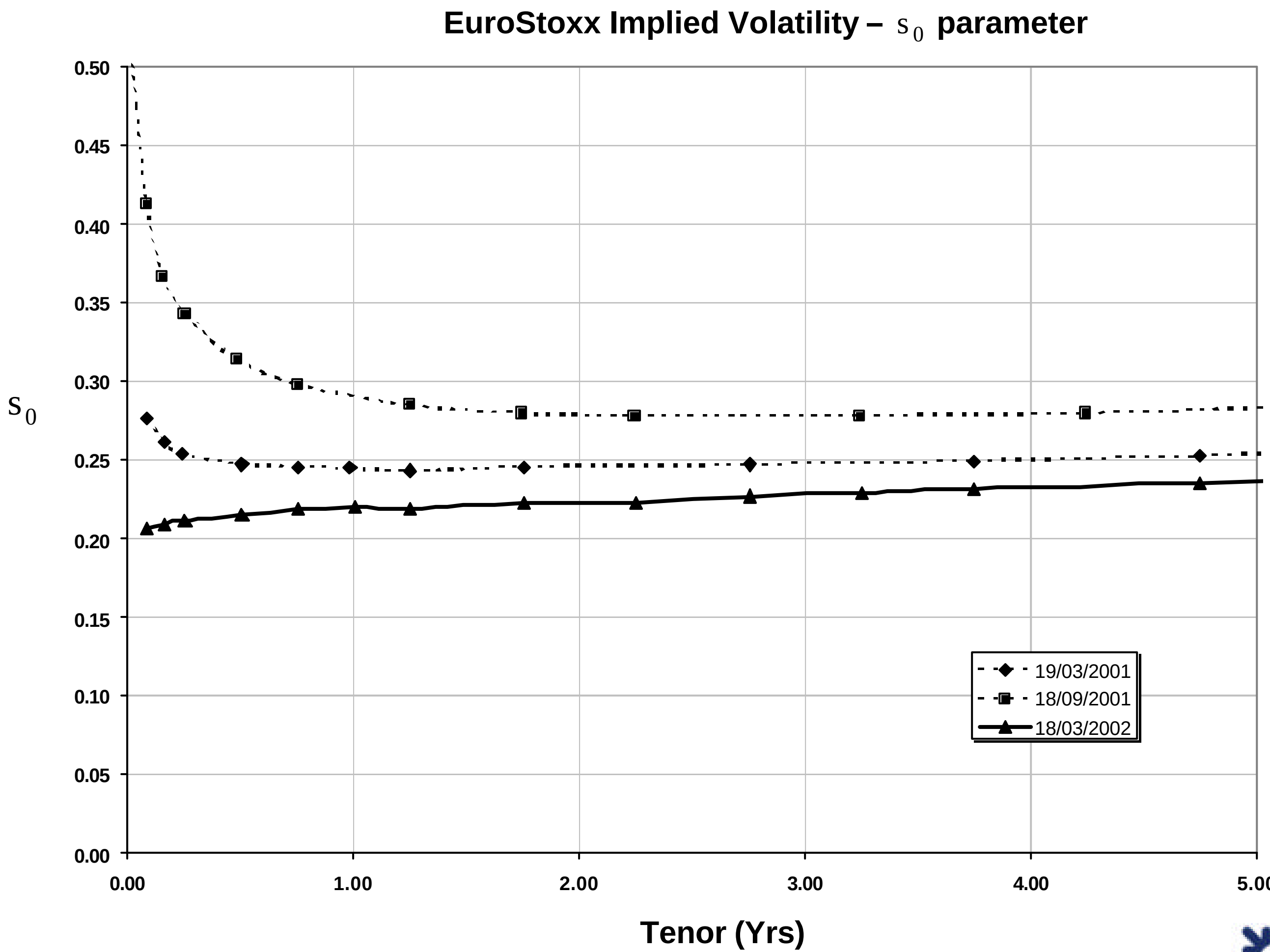

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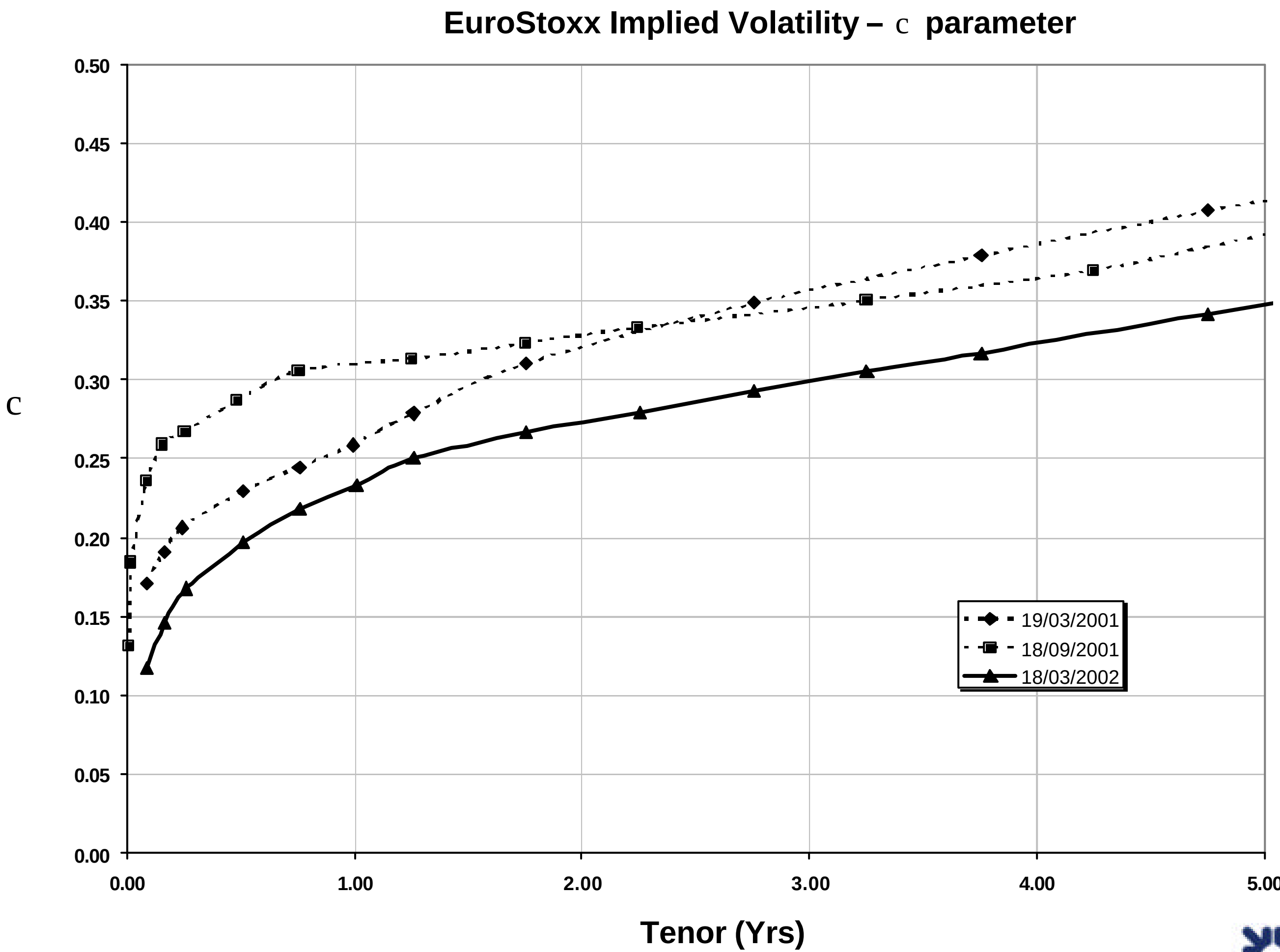

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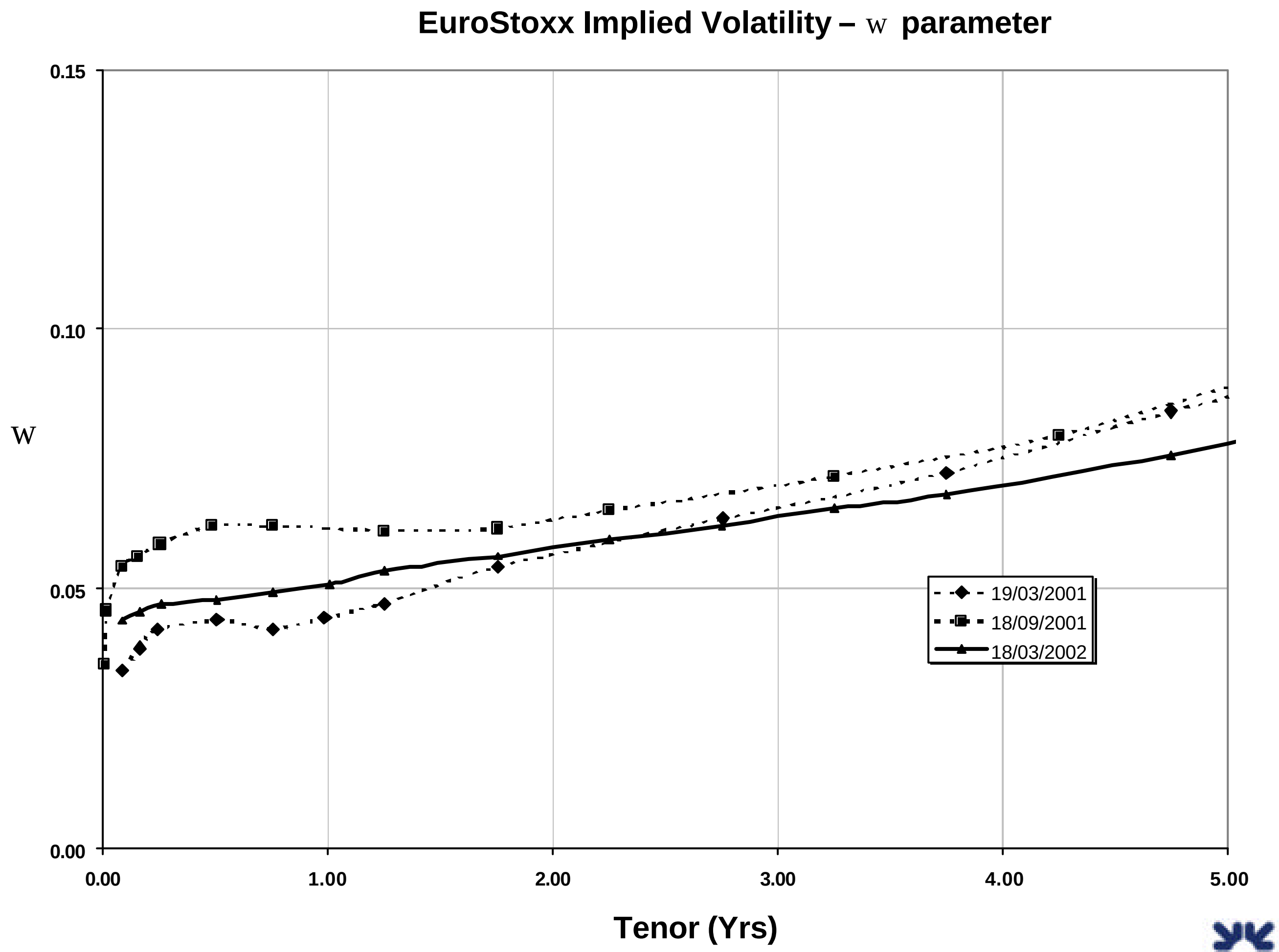

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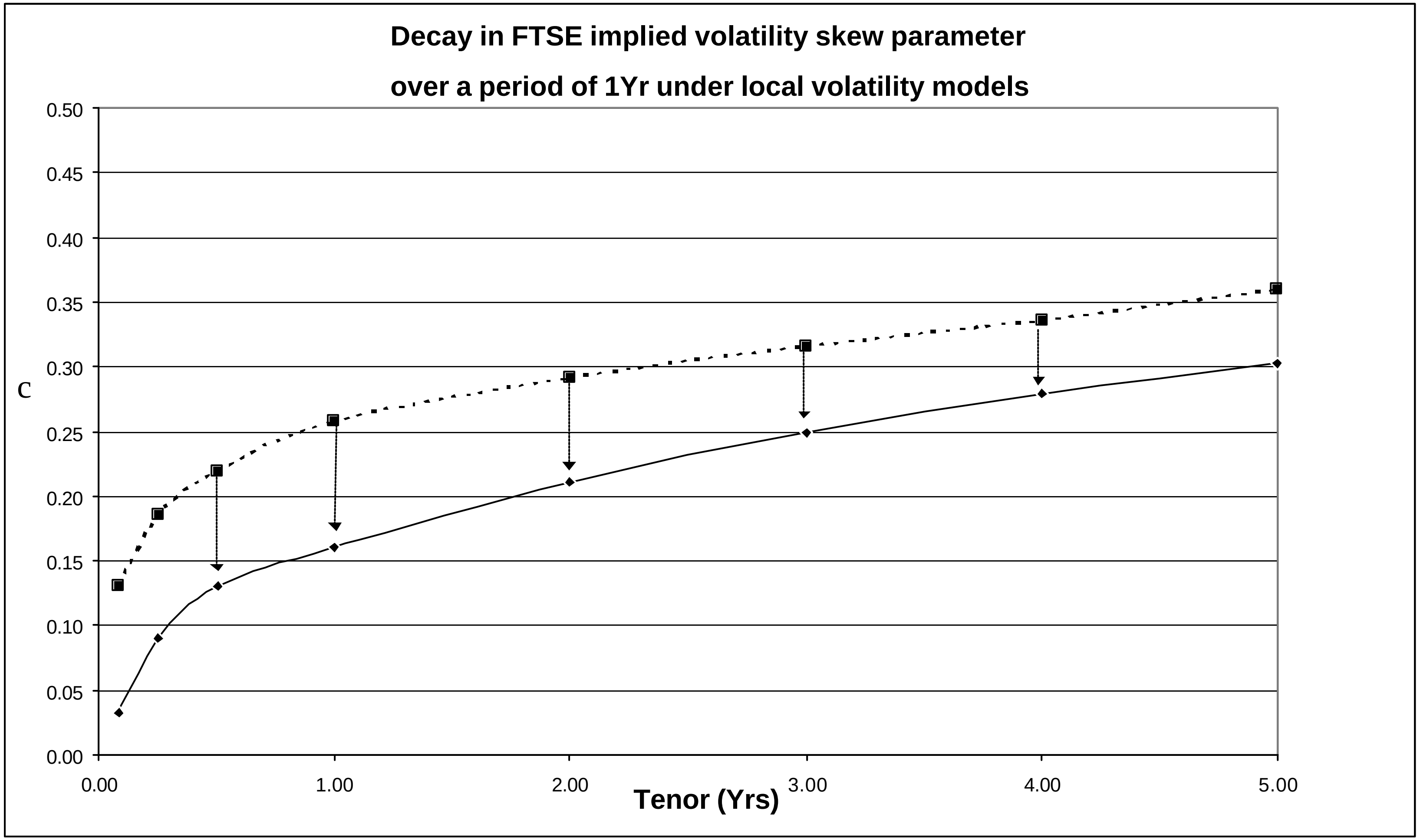

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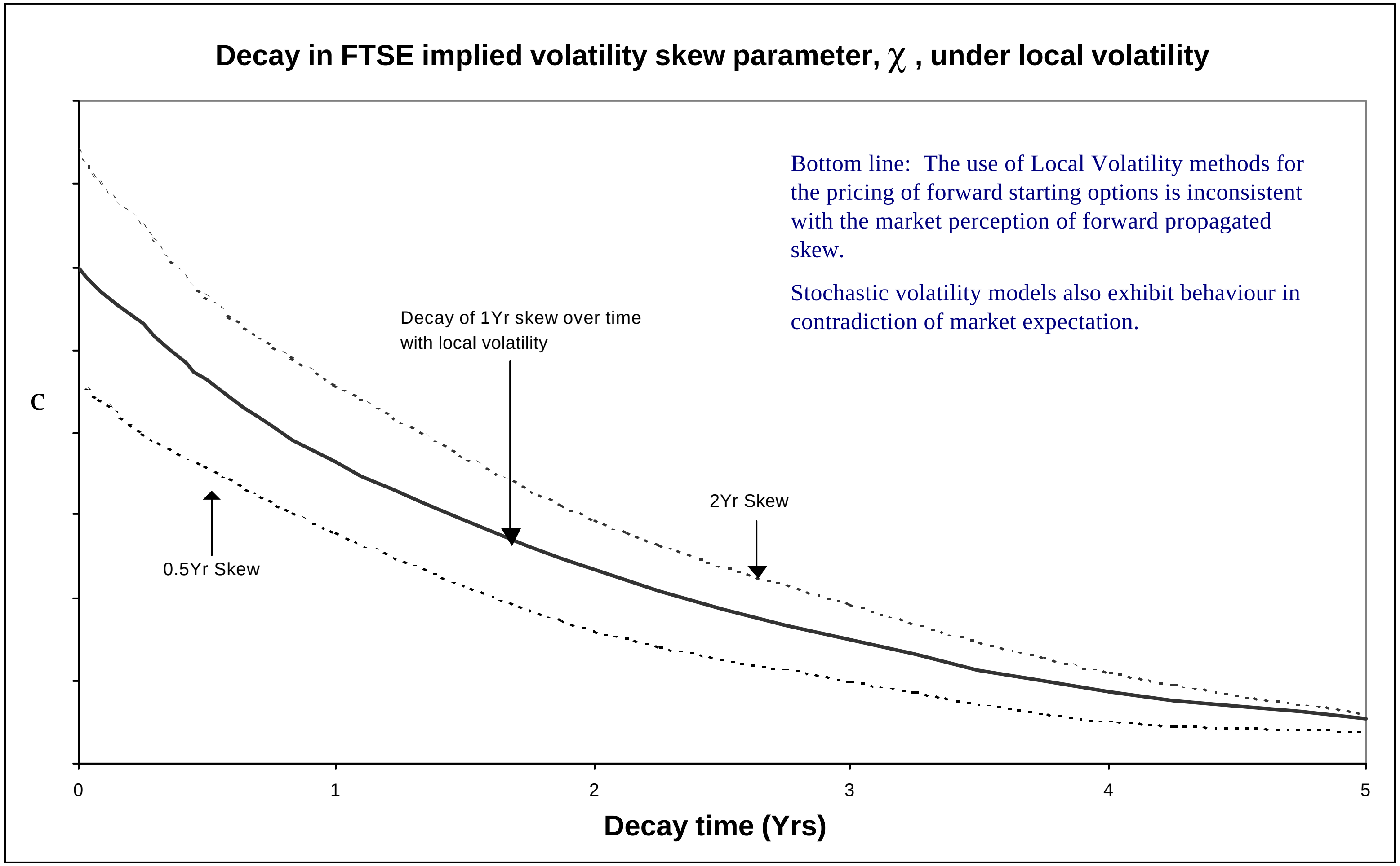

Pricing Forward Starting Options (from D Samuel)

An alternative modelling approach? …

Assume that the volatility surface can be characterised at a given time by the parametric curves: $(\sigma_0 , \chi , \omega)$,

Assume that the time evolution of these curves is stochastic, but characterised by some simple normal modes and a low number of Weiner processes (currently giving consideration to one and two-factor models).

Thus, if we had an interest in pricing a 1-Yr option starting in two years then we would need to evolve $(\sigma_0[1] , \chi [1] , \omega [1])$ two years under their respective SDE's, and then evaluate the expectation value of the 1-Yr option under their respective distributions.

Assumptions and Set-Up

- Perfect market
- Traded Instruments: Calls [continuum of strikes + discrete maturities]+ Complex Product
- Price Information: underlying + all calls today
- Pricing Condition: existence of pricing measure today
- Definition of state of the world: underlying + prices of all calls and puts

Admissible Smile Surfaces

Definition (ADMISSIBLE SMILE SURFACE) *Define a smile surface such that the associated call prices satisfy*

$$\frac{\partial Call(t,T,S_t,T)}{\partial K} < 0 \quad \#$$

$$\frac{\partial^2 Call(t,T,S_t,T)}{\partial K^2} > 0 \quad \#$$

$$\frac{\partial Call(t,T,S_t,T)}{\partial T} > 0 \quad \#$$

$$\frac{\partial Put(t,T,S_0,T)}{\partial K} > 0 \quad \#$$

$$Call(t,T,S_t,T)|_{K=0} = S_t \quad \#$$

$$lim_{K\to\infty} Call(t,T,S_t,K) = 0 \quad \#$$

an ***admissible smile surface****.*

Admissible Smile Surfaces

Admissible smile surfaces prevent the possibility that, for instance, more out-of-the-money calls should be worth more than more-in-the-money calls; or require a strictly positive price density. For today's smile surface, the admissibility conditions are necessary and sufficient in order to rule out the possibility of static strategies **constructed today** that can be arbitraged

When the smile surface in question is in the **future**, however, I shall show that the admissibility conditions are **necessary**, but **not sufficient** for absence of model-independent arbitrage.

Present and Future Smile Surfaces

One can establish a one-to-one correspondence between the current set of prices, the current smile surface and the price density.

$$f_0(S_T) = \frac{\partial^2 Call_0(S_0, K, t_0, T)}{\partial K^2} = \frac{\partial^2 BS(S_0, K, t_0, T, \sigma_{impl}(t, T, K, S_0))}{\partial K^2} \quad \text{\# (Eq1)}$$

The same relationship applies in the future (and the price density becomes conditional).

Conditional State Densities

Definition (FUTURE CONDITIONAL RISK-NEUTRAL STATE DENSITY) *The* ***future (time–t) conditional risk-neutral state density,*** $\Phi_t(X_T|X_t)$, *is defined to be the risk-neutral probability density that the world will be in state X_T at time T given that state X_t prevails at time t.*

Remark *The state of the world at time t can be equivalently described in terms of*

i) the value at time t of the underlying plus the values of all the calls:

$$X_t = \{S(t) \cup Call_t(S_t, K, t, T), \forall K, T\}$$ #

ii) the value at time t of the underlying and the associated future (time-t) S_t-conditional implied volatility surface:

$$X_t = \{S_t \cup \sigma_{impl}(t, T, K, S_t), \forall K, T\}$$ #

iii) the value at time t of the underlying and the associated future (time-t) S_t-conditional risk-neutral density:

$$X_t = \{S_t \cup f_t(S_T|S_t)\} \qquad \forall T$$ #

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Price and State Densities Compared

- In general the future probability density of the stock price is not enough to determine the future prices of calls (eg, stochasticvolatility process)
- The state density contains much more information than the price density
- Under what conditions can one derive the state density from the price density?

Deterministic Smiles

Given an admissible smile surface today,a *future*, time-t *conditional* smile surface is said to be ***deterministic*** if the future smile surface can be expressed as a deterministic function of time, maturity, strike of the realization of the stock price at time t.

Examples of deterministic smile surfaces:

- geometric-diffusion (Black-and-Scholes) process with constant and time dependent volatilities;
- jump diffusion with constant or time-dependent coefficients;
- displaced diffusions (Rubinstein (1983)) and their generalizations such as displaced jump-diffusions;
- Derman-Kani restricted-stochastic (local) volatility model;
- Variance Gamma process; etc.

Consequences of Deterministic Smile Surfaces

If the future smile surface is deterministic, given the knowledge of the future value of S_t the prices of all future calls are also known. Therefore, the state X_t is fully determined by S_t, and the (conditional and unconditional) price and state densities coincide:

$$\Phi(X_t) = f(S_t) \quad \#$$

$$\Phi(X_T|X_t) = f(S_T|S_t) \quad \#$$

Deterministic smile surfaces are important because they allow us to work with the much simpler *price* densities rather than the state densitites. For any current price density, $f_0(S_0)$, and conditional deterministic price density, $f_t(S_T|S_t)$, it is always true that

$$f_0(S_{T_2}) = \int f_0(S_{T_1}) f_{T_1}(S_{T_2}|S_{T_1}) dS_{T_1} \quad \#$$

Using Equation ref: Eq1 one can therefore write:

$$f_0(S_{T_2}) = \frac{\partial^2 BS(S_0, K, t_0, T_2, \sigma(t_0, T_2, S_0, K))}{\partial K^2} \quad \text{\# (Eq4)}$$

$$f_0(S_{T_1}) = \frac{\partial^2 BS(S_0, K, t_0, T_1, \sigma(t_0, T_1, S_0, K))}{\partial K^2} \quad \text{\# (Eq5)}$$

$$f_{T_1}(S_{T_2}|S_{T_1}) = \frac{\partial^2 BS(S_0, K, T_1, T_2, \sigma(T_1, T_2, S_{T_1}, K))}{\partial K^2} \quad \text{\# (Eq6)}$$

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Kolmogorov-Compatible Smile Surfaces

In order to lighten notation, denote the operator

$$\frac{\partial^2 BS}{\partial K^2}[\;] \equiv \Theta[\;] \qquad \text{#}$$

Then Equation ref: Eq6 can be re-written as

$$\Theta(S_0, K, t_0, T_2, \sigma(t_0, T_2, S_0, K)) = \qquad \text{# (Eq3)}$$

$$\int \Theta(S_0, K, t_0, T_1, \sigma(t_0, T_1, S_0, K)) \Theta(S_t, K, T_1, T_2, \sigma(T_1, T_2, S_{T_1}, K)) dS_{T_1}$$

with $\Theta(S_0, K, t_0, T_1, \sigma(t_0, T_1, S_0, K))$ and $\Theta(S_0, K, t_0, T_2, \sigma(t_0, T_2, S_0, K))$ market-given, and $\Theta(S_t, K, T_1, T_2, \sigma(T_1, T_2, S_{T_1}, K))$ is to be determined so as to satisfy Equation ref: Eq3.

Remark *There is a one-to-one correspondence between the quantity*

$$\Theta(S_t, K, T_1, T_2, \sigma(T_1, T_2, S_{T_1}, K))$$

and future conditional deterministic densities (conditional future smile surfaces). There is in general an infinity of solutions

$$\Theta(S_t, K, T_1, T_2, \sigma(T_1, T_2, S_{T_1}, K))$$

such that Equation ref: Eq3 is satisfied. Therefore, even if we require the smile surface to be deterministic,

there still exists an infinity of future smile surfaces compatible with today's prices of calls and puts.

Definition (KOLMOGOROV COMPATIBILITY) *Define any future deterministic conditional density or smile surface such that Equation ref: Eq3 is satisfied a* ***Kolmogorov-compatible density****.*

Necessary Condition for Absence of Model-Independent Arbitrage

- Given a current admissible smile surface, if all the future deterministic smile surfaces for times T1,T2,...,T_n are Kolmogorov compatible no model-independent strategy revised on the same set of dates can generate arbitrage profits.
- The trader's dream revisited

Conditions for Uniqueness of Kolmogorov-Compatible Surfaces

- The equations obtained up to this point determine the links between the present and the future densities that must be satisfied by deterministic smile surfaces in order to avoid model-independent arbitrage. One extra condition is required in order to ensure uniqueness of the resulting conditional density. This condition is often implicitly assigned by popular process-based models, but can be stated explicitly in the present set-up

The Distance Condition

Condition (DISTANCE CONDITION) *Let us assume that a Kolmogorov-compatible conditional probability density is of the form*

$$f(S_T|S_t) = f'(P(S_T) - P(S_t)) \qquad \text{# (Eq7)}$$

for some functions $f'()$ and $P()$. If this is the case, the probability density is said to satisfy the distance condition.

Equation ref: Eq7 requires that the transition probability of the stock price at two different times should only depend on the distance between (some function of) the starting and arrival points (whence the name 'Distance Condition'). If the function $P(S) = S$ one recovers a normal diffusion for the underlying. If $P(S) = ln(S)$ one recovers a log-normal diffusion. If $P(S) = S + \alpha$ we are in the displaced-diffusion case; etc.

Remark *The current price density can always be written as some function f'_0 of $P(S_0)$. If the [DISTANCE CONDITION] is satisfied, the current risk-neutral density for the function P of the underlying for time T_2, f', can be written as a convolution:*

$$f'_0(P_{T_2}) = \int f'_0(P_{T_1}) f'(P_{T_2} - P_{T_1}) dP_{T_1} =$$

$$= f'_0(P_{T_1}) * f'_0(P_{T_2}) \qquad \text{#}$$

where the symbol $*$ indicates convolution, and, to lighten notation, we have denoted by P_{T_1} the quantity $P(S_{T_1})$.

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From the Distance to the Uniqueness Condition

Proposition *Denote by $\mathcal{F}$ and $\mathcal{F}^{-1}$ the Fourier and the inverse Fourier transform operators, respectively. Then, given times T_1 and T_2, if a soultion exists, there is a unique future deterministic time-T_1 conditional density (smile surface) for expiry at time T_2 compatible with today's state of the world/ (with todays' prices for plain-vailla calls and puts)/(with today's smile surface). It is given by*

$$f'_{T_1}(P_{T_2} - P_{T_1}) = \mathcal{F}^{-1}\left[\frac{\mathcal{F}[f'_0(P_{T_2})]}{\mathcal{F}[f'_0(P_{T_1})]}\right]$$

#

Therefore, under the [DISTANCE CONDITION], the future, conditional risk-neutral density, and, therefore, the future conditional smile surface can be obtained from the market-given risk neutral densities. More precisely, if the [DISTANCE CONDITION] is satisfied, for a given P, if a solution exists it is unique, i.e. there exists a unique Kolmogorov-compatible future density.

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Homogeneity Conditions

One of the potentially desirable conditions for a smile function is that it should be self-similar when the its arguments S{t} and t undergo certain transformations. In particular, we can ask the following questions:

- What will the smile surface look like **when the underlying changes?**
- What will the smile surface look like **when we move forward in time?**

The answer to the the first question leads to the concept of floating or sticky smiles. The second question is related to the existence or otherwise of an arbitrage-free forward-propagated smile.

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Floating Smiles

The smile surface today, i.e. for a fixed S_0, can always be written as a function, $\hat{\sigma}()$, of $ln[K/S_0] \equiv y_0$:

$$\sigma_{impl}(t_0, T, K, S_0) = \hat{\sigma}(t_0, T, ln[K/S_0]) = \hat{\sigma}(t_0, T, y_0)$$ #

This observation is useful in establishing the following conditions, which are central to the treatment to follow.

Condition (STOCK HOMOGENEITY) *Let us impose that the time-t smile surface is deterministic and that it should be of the form*

$$\sigma_{impl}(t, T, K, S_t) = \hat{\sigma}(t, T, y_t)$$ #

with $y_t \equiv ln[\frac{K}{S_t}]$ *and* $\hat{\sigma}$ *the same function that describes the current smile surface.*

Definition (FLOATING SMILE) *A future deterministic smile surface such that Condition [STOCK HOMOGENEITY] is satisfied for all t is called a **floating smile surface**.*

Since the definition of deterministic smile surface requires that Condition [STOCK HOMOGENEITY] should hold for any t, it must be true also for an instantaneous change in the stock price. This condition therefore directly relates to the translation properties (in log space) of the smile surface with the stock price.

Arbitrage-Free Floating Surface

However, we do not know yet whether, and under which conditions, such a floating surface can exist without allowing arbitrage. This would certainly be the case if there were a process that produces deterministic floating smile surfaces, but we have not based our treatment on the specification of a particular process, and we must, therefore, follow some other route. In our language, the condition necessary for the existence of a deterministic floating smile is the following:

Proposition *Let us assume that Conditions [STOCK HOMOGENEITY] and [DISTANCE CONDITION] are satisfied. If the conditional probability density is of the form* $f(S_T|S_t) = \xi(lnS_T - lnS_t)$*, i.e. if the function* $P()$ *in remark* 21 *and Proposition* 1 *is given by* $P \equiv ln(S)$ *(and* f' *is therefore the probability density for* $ln[S_t]$*), then the corresponding future smile surface is floating.*

Remark *All pairs* $\{K, S_t\}$ *such that their ratio is a constant (and for fixed* t *and* T*) produce the same value for* y_t*. Therefore, if the smile surface is floating, all such pairs* $\{K, S_0\}$ *give rise to the same implied volatility, and a set of call prices simply proportional to* S_t *(given the homogeneity properties of the Black and Scholes formula).*

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Forward-Propagated Smile Surfaces

Definition (FORWARD-PROPAGATED SMILE) *A floating smile surface such that the function that gives the implied volatility a a function of rsidual maturity and strike is independent of calendare time is said to be forward-propagated*

Remark *There is no guarantee, in general, that a forward-propagated smile will be Kolmogorov-compatible (ie, that today's prices admit a forward-propagated smile without allowing model-independent arbitrage opportunities). If the trader felt that forward-propagation were a desirable property, she could try to find the future condition densities (smile surfaces) that are Kolomogorov-compatible, and that are 'closest' - given some suitably defined distance - to forward-propagated densities (smile surfaces)*

Summarizing:

- **<u>Deterministic smile</u>**: There exists some function of **maturity** T, **strike** K and **time** t such that, *conditional on the future stock price being known*, the future smile surface is known today exaclty.
- **<u>Floating smile</u>**: There exists some function today of **time** t, **maturity** T and of the **ratio** y{t}=K/(S{t} in terms of which one can express today all the future smile surfaces
- **<u>Forward-propagated smile</u>**: There exists some function today of residual **maturity** T and of the **ratio** y{t}=K/(S_{t} in terms of which one can express today all the future smile surfaces

Stochastic Smiles

We intend to describe the stochastc evolution of smiles, by describing today's smile as a function of a number of parameters, {a}, and by assigning a stochastic behaviour to these parameter.

This is done as follows.

Condition (STOCHASTICITY) *Let us impose that the future (time-t) conditional implied volatility function, $\sigma_{impl}(t,T,K,S_t)$, should be a stochastic quantity, whose values depend on the realization of a discrete set of random variables $\{\alpha_t\}$.*

Condition (DISCRETENESS) *Let us assume that the random variables $\{\alpha\}$ that determine the realization of the future implied volatility surface (the future conditional density) can assume an arbitrary large but finite number of values. Let $\{\pi_{ij}^t\}$ denote the probability of i-th realization of the j-th parameter α at time t.*

Remark *The approach is superficially similar to that employed by Joshi and Rebonato (2001) in their stochastic-volatility extension of the LIBOR market model. The important difference is that we assume in this work that the parameters describing the* ***implied*** *volatility surface are stochastic. Joshi and Rebonato (2001), on the other hand, assume that the parameters of the* ***instantaneous*** *volatility are stochastic. This apparently minor difference ensures automatically that all the resulting future smile surfaces are Kolmogorov-compatible, and arbitrage free.*

Condition (INDEPENDENCE) *Let us assume that future smile surface can be written as $\sigma_{impl}^t = \sigma(t,T,y_t;\{\alpha_t\})$ and that the values of these random variables α at time t should be independent of $y(t)$:*

$$Prob(\alpha_t|y_t) = Prob(\alpha_t) \rightarrow Prob(\alpha_t,y_t) = Prob(\alpha_t)Prob(y_t)$$

#

The Trader's Dream Revisited

- As for Condition [STOCHASTICITY], the random variables {a} could be of very different nature: they could, for instance, be the second, third, fourth, etc. moments of a future probability density; they could be the future market prices of at-the-money volatilities, straddles and risk reversals; they could be the coefficients of a parametrically fitted density (see, e.g. Mirferndereski and Rebonato (2001), Samuel (2002)). All these interpretations are possible, as long as the random variables, however chosen, are independent of y{t}.

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Introducing Equivalent Deterministic Smile Surface

Remark *If no arbitrage is to be allowed, a probability measure must exist such that the relative price of a call today is given by the weighted expectation of the relative call price (payoff) at time t. Let us assume that we are dealing with a stochatsic smile surface such that conditions [INDEPENDENCE] and [STOCHASTICITY] are satisfied. Then, if the numeraire is chosen to be the (deterministic) discount bond maturing at time* T, $Z(0,T)$, *one can write, for* $t \leq T$,

$$\frac{Call_0(S_0,T)}{Z(0,t)} = Z(t,T)E_P[Call_t|\Im_{S,\alpha_0}] =$$

$$Z(t,T)\sum \pi_i \int Call_t(S_t,T,X_i(t))f_0(S_t)dS_t =$$

$$= Z(t,T)\sum \pi_i \int BS(S_t,t,T,\sigma_{impl}(t,T,S_t,K;X_i(t))f_0(S_t)dS_t =$$

$$= Z(t,T)\sum \pi_i \int BS(S_t,t,T,\hat{\sigma}_{\alpha_i}(t,T,y_t))f_0(S_t)dS_t =$$

$$= Z(t,T)\sum \pi_i \int [\int (S_T - K)^+ f_t(S_T|S_t)_i dS_T] f_0(S_t)dS_t \qquad \text{\# (Eq9)}$$

where

π_i *is the probability of the i-th realization of the multiplet* $\{\alpha\}$. Note carefully that the quantity y_t depends on the strike, K.

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Equivalent Deterministic Smile Surface

If the smile is floating, the price of a call today in the presence of a stochastic floating smile is identical to the price that would obtain with the single deterministic stock-homogeneous (floating) future smile associated with the average conditional density $\overline{f_t}(S_T|S_t)$. *Such as smile is called the* ***equivalent deterministic future smile***.

Proof From Equation ref: Eq9, after interchanging the order of integration one obtains:

$$Call_0(S_0,T)/Z(0,T) =$$

$$= \sum \pi_i \int [\int (S_T - K)^+ f_t(S_T|S_t)_i dS_T] f_0(S_t) dS_t =$$

$$= \int [\sum \pi_i \int (S_T - K)^+ f_t(S_T|S_t)_i dS_T] f_0(S_t) dS_t =$$

$$= \int [\int (S_T - K)^+ \sum \pi_i f_t(S_T|S_t)_i dS_T] f_0(S_t) dS_t =$$

$$= \int [\int (S_T - K)^+ \overline{f_t}(S_T|S_t) dS_T] f_0(S_t) dS_t =$$

$$= \int BS(S_t, t, T, \overline{\sigma}(t,T,y_t)) f_0(S_t) dS_t$$

#

where $\overline{f_t}(S_T|S_t)$ is the average conditional density, and $\overline{\sigma}(t,T,y_t)$ the associated deterministic floating smile (implied volatility).

Implications of the Existence of an Equivalent Deterministic Smile Surface

A variety of processes have been proposed in order to describe the stock price dynamics. Each of these processes gives rise to a set of future smile surfaces. In some cases these smile surfaces are deterministic, in other stochastic.

A stochastic-smile-surface stock price process will produce prices for calls and puts different from the prices from the equivalent deterministic future smile only to the extent that it produces stochastic future smiles which are *not* independent of the future realization of the stock price.

Extension to Displaced Diffusions

- Empirical observations indicate that there exists a negative correlation between the future level of smile surfaces and of stock prices. This would seem to invalidate one of the crucial conditions of the approach outlined above.
- If the dependence is relatively simple, however, the approach can sometimes be rescued by a simple change of variables. One possible way to do this is to recast the distance condition in terms of a function other than $\ln(S_{t})$. Another attractive route is to employ the approximate but accurate equivalence between CEV process and displaced-diffusion processes (Marris, Rubinstein).

The Path to Displaced Diffusions

Logical Steps:

- Restrict local volatility functions to power-law dependence of the volatility on the stock price (CEV models)
- Exploit the approximate but accurate equivalence between CEV processes and displaced diffusion processes

Recasting the treatment in a DD framework

Definition *Define the present or future a-displaced implied volatility,* σ^a_{impl}*, as the quantity that, input in the Black-and-Scholes formula with* $(S + a)$ *as spot and* $(K + a)$ *as strike produces the value of a call with spot equal to* S *and strike equal to* K*. Also, define*

$$y^a_t \equiv ln[\frac{K + a}{S_t + a}]$$

#

Mutatis mutandis, the treatment presented above can be re-cast in terms of the new quantity y^a_t. Clearly, the distance condition is now expressed in terms of the function $\ln(S_t + a)$.

Limitations of the Approach

Consider stochastic-volatility process with the innovation in the volatility independent of the innovation in the stock price

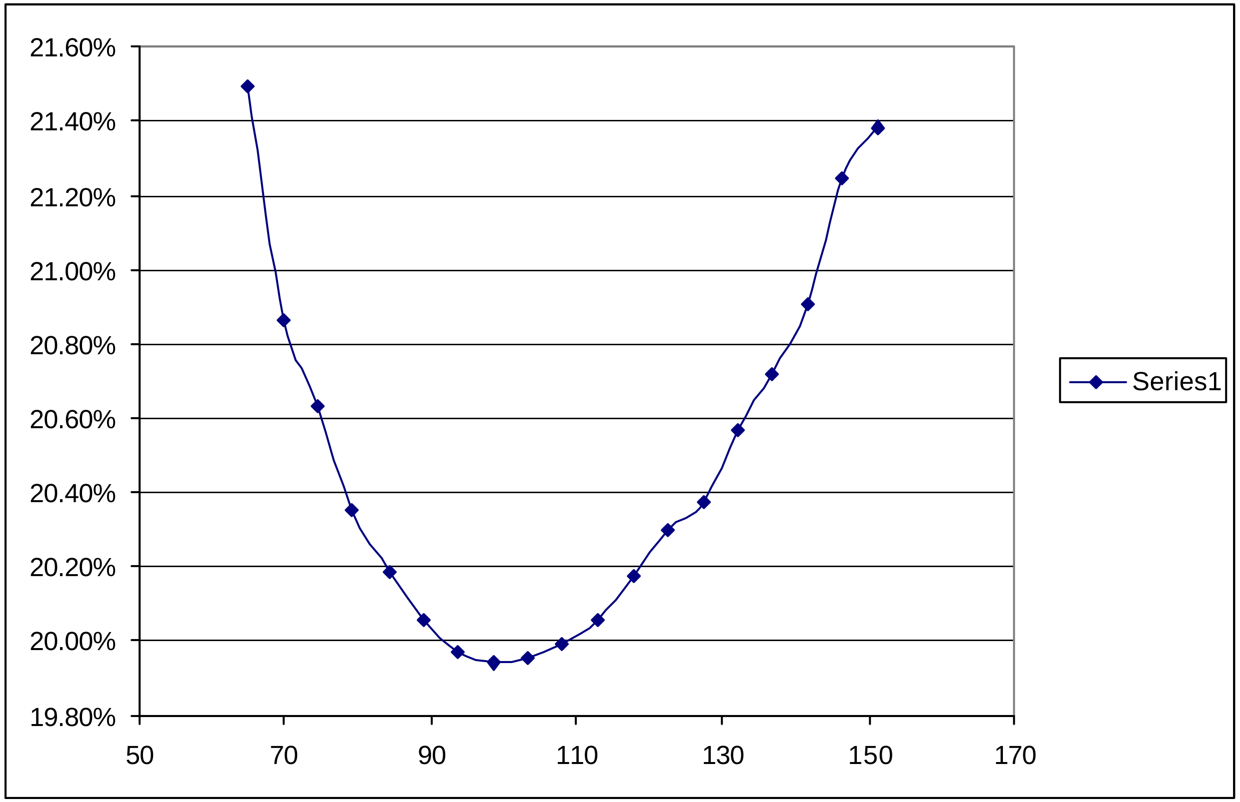
Assume that at time T the stock price is ‘much higher’ (lower) than its value today

What is the likelihood that the volatility at time T is also very high?

What does this imply for

- the level of the future smile surface
- the slope of the future smile surface?

Conditional expectation of future volatility as a function of future realization of stock price



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