Draft:Assigning Future Smile Surfaces: Conditions for Uniqueness and Absence of Arbitrage

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Abstract

We explore in this paper to waht extent future implied volatility surfaces can be assigned by a trader without incurring the risk of model-independndt arbitrage. We give precise meaning to the concepts of floating and forward-propagated smile, and show under what condtions such smiles can be arbitrage free. We introduce a mechanism to produce future stochastic smile surfaces. We show that, if some conditions are met, for pricing purposes a future deterministic smile surface would produce the same prices produced under a stochastic setting.

1 Introduction

1.1 Motivation and Plan of the Work

We present in this paper some results regarding the model-independent, arbitrage-free specification of future volatility smiles. The specification of an arbitrage-free dynamics for the smile surface is not a novel idea, see e.g. Schoenbucker (1998). The crucial difference between his approach and ours, however, is that Schoenbucher posits a particular (stochastic volatility) process for the underlying and derives, *given this process*, the arbitrage-free dynamics for the implied volatility surface. In this work we do not assume the process for the underlying to be known. Our results are therefore weaker, but more general.

We believe that our work can be both of practical and conceptual relevance to traders and risk managers. To begin with, traders directly observe the evolution of smile surfaces. The reasonableness of a process-based model is mainly assessed on the basis of its ability i) to reproduce today's smile surface and ii) to produce future smile surfaces consistent with the trader's beliefs,intuition and trading views. Indeed, as Britten-Jones and Neuberger (1998) point out, even if one restrict the analysis to purely continuos processes, one can obtain a perfect match to an arbitrary set of *current* call and put prices (i.e. one can match *today's* smile surface perfectly) with an infinity of, say, stochastic-volatility processes. Since the ability to fit a current smile surface cannot by itself discriminate between 'good' and 'bad' models, other criteria of 'goodness' must be found. In this context, it is natural for the trader to think in terms not only of present, but also of future, smile surfaces and to bring her financial views about future smiles to bear as much as possible on the choice of the most appropriate process-based model. Arguably, the 'underlying' for the option trader is the future volatility surface, and it is therefore appropriate and desirable to work within a framework that allows the trader to express views directly about the quantities in which she, directly or indirectly, makes a market.

A practice often emplyed by traders, especially when hedging is complex, is to by-pass a process-based modelling stage, and directly prescribe a future smile surface to be either 'identical' to today's smile, or modified according to the trader's views. See, e.g. Samuel (2002). We show that this forward-propagated smile is, in general, not compatible with absence of arbitrage, even if the market has no knowledge of the true process for the underlying.

Another area of work where the ability to assign future smiles is very important is pricing by dynamic replication. With this well known approach (see,e.g., References here) a portfolio of plain-vanilla calls is built today, such that all the initial and boundary conditions of a complex poduct are satisfied at all future times and in all states of the world. In general, this strategy will entail unwinding positions in the plain-vanilla calls at a future point in time. The future cost of doing so is not known in a model-independent way today, and depends on the realization of a future smile. The trader might well like to prescribe future smile surfaces congruent with his views or expectations. Unfortunately, given the prices of the plain-vanilla options today, future smile surfaces cannot be arbitrarily assigned without incurring the risk of model-independent arbitrage. Therefore some of results presented in this work clarify what conditions must be satisfied by user-assigned future smile surfaces to ensure that no true-porcess-independent strategies can aribtrage the resulting prices (unless, as discussed below, the arbitrageur has superior knowledge about the true process).

Another concept that traders are familair with is the distinction between 'sticky' or 'floating[1]' smiles. Also, traders often speak and think in terms of 'forward-propagated' smiles. These concepts have been criticized as being rough rules of thumb, without theoretical foundations (see,.e.g. Derman (1999)). We show that the concepts can be made precise and defined in such a way that they reflects traders' usage and intuition. The price to be paid, however, in terms of restriction on the dynamics of the smile surface, is non-negliglible. We therefore provide a precise definition of a floating smile, we show under what conditions today's prices allow for the existence of such a floating smile surface, and we

[1] The term 'sticky-delta' is often used to described floating smiles. The discussion in section 2 shows why, and in what circumstances, this two terms can be used interchangeably.

highlight the implicit assumptions a trader makes when speaking, for instance, of a floating or of a forward-propagated smile. We also show that, if it exists, the future floating smile surface consistent with today's price is unique, and can, in principle, be obtained by Fourier transform.

An unexpected by-product of our analysis is that, under relatively mild conditions, even if the future smiles are stochastic, as long as they are independent of the future realization of the underlying, there exists a single future **deterministic** floating smile such that options have exactly the same price if valued under the stochastic regime or in this equivalent deterministic setting.

We conclude our paper by presenting some computational/empirical results showing to what extent popular processes statisfy the independence condition mentioned above. In particular, we show that if the underlying true process were a jump-diffusion with stochastic intensity and/or jump amplitude and/or variance of amplitude, the future smile surfaces display a very mild dependence on the future realization of the underlying. We conclude that, in the case of this process, introducing stochasticity to the jump component 'buys' the trader very little on top of the results given by the much simpler equivalent deterministic setting, and, therefore, the replication strategy proposed below can beapplied in an approximate but accurate way also if the underlying process were a jump-diffusion with stochastic jump amplitude and frequency. We also find that, if the true process were, instead, a stochastic-volatility diffusion there would be an appreciable correlation between future smile surfaces and future realizations of the underlying, and that therefore stochastic-volatility processes do not produce prices that could be obtained from a set of equivalent future smile surfaces.

1.2 Relevance of Our Work

We believe that these findings have a rather wide conceptual interest. To begin with, we show that the only additional feature of a stochastic-parameter[2] model not provided by a suitable associated deterministic version that has any bearing on pricing is the ability to create future smile surfaces which display a dependence on future realizations of the underlying. Indeed, we show that in some cases, such as jump-diffusions with stochastic parameters, a rather complex set up provides the user with very little, in terms of pricing and hedging, that a well-chosen jump-diffusion process with deterministic parameters could not offer. This observation suggests that a perturbative approach around the deterministic solution could be attractive. We also show that, as long as the probability for the stock to move from value S_{T_1} and time T_1 to value S_{T_2} at time T_2 only depends on $g(S_{T_1}) - g(S_{T_2})$, with $g(\cdot)$ a known arbitrary function of S, the future probability density and the future smile surface are uniquely determined by today's prices of calls expiring at times T_1 and T_2.

[2]such as, for instance, a stochastic-volatility model, or a jump-diffusion model with stochastic jump intensity and/or jump amplitude and/or expectation of the jump amplitude ratio.

At a more practical level, as mentioned above and clearly highlighted by Britten-Jones and Neuberger (1998), more and more approaches have been recently proposed in the literature that allow a perfect or almost perfect fit to the current smile surface [references here Andersen and Andreasen]. Despite the very high, and similar, quality of the cross-sectional fit to current prices, these models offer very different descriptions of the underlying dynamics, and can therefore produce different prices and suggest different hedging ratios for exotic options. We suggest that an emprical observation of the actual behaviour of smiles as a function of changes in the underlying can provide useful indications as to the most appropriate model, and therefore supply the trader with a criterion for choosing between competing, and similarly well-fitting, candidate processes. We are aware that, if the volatility is non-determinstic, care must be taken in making comparisons between volatility surfaces in the real-world and in the risk-neutral measure, but we concur, again, with Britten-Jones and Neuberger that understanding any systematic differences between realized and implied volatilities is central to understanding the pricing of volatility risk. Furthermore, despite the fact that the statistically-observed and the model-implied changes in smile surfaces inhabit different probability measures (the objective and the risk-adjusted one, respectively) Rebonato and Joshi (2001) have recently introduced techniques to carry out a meaningful comparison between the two quantities in the interest-rate world.

2 Establishing the Conditions of No-Arbitrage for the Stochastic Evolution of Future Smile Surfaces

2.1 Description of the Market

Condition 1 (PERFECT MARKET) *We place ourselves in an economy with a perfect, friction-less market, where traders incur no bid/offer spreads, short sales of calls and puts are allowed in arbitrarily large (but finite) sizes, no taxes are levied, etc.*

Condition 2 (TRADED INSTRUMENTS) *In this perfect-market economy, an underlying 'stock' and plain-vanilla calls of all maturities and strikes are traded. The strikes span a continuum of values, but, for the sake of simplicity, the maturities of the plain-vanilla calls and puts belong to a set* $[T_i], i = 0, 1, 2, ..., N$*, with* N *arbitrarily large but finite.* T_0 *represents today. Also for the sake of simplicity, we shall assume deterministic interest rates, and therefore* N *deterministic bonds,* $B_i, i = 1, 2, ..., N$*, also trade. Finally, trading also takes place in complex products of contractual maturity* $T_k \in [T_i]$*, i.e, in instruments whose payoffs may depend on the history of the underlying up to and including time* T_k*.*

Condition 3 (PROBABILITY SPACE) *We place ourselves in a filtered probability space* $(\Omega, \mathcal{F}_t, Q)$*. The state space* Ω *that we require contains all the present and possible future realizations of the underlying and of the prices of plain-vanilla calls at times* $[T_i]$*.* $\mathcal{F}_t$ *is the natural filtration generated by the prices of all the plain-vanilla calls and of the underlying on the arbitrarily large but finite number,* $N+1$*, of dates* $[T_i]$*.*

By availing ourselves of the knowledge of the filtration $\mathcal{F}_t$ we require that we can know, at each of the discrete points in time, what smile surfaces and which values of the underlying have occurred, and which have not. As for the probability measure Q, it is characterized by the following condition:

Condition 4 (PRICING CONDITION) *If we denote by* $C(K, T_0, T_j)$ *the price today of a* K*-strike call of maturity* T_j*, and by* S_{T_j} *the value of the stock at time* T_j*, then we require that a measure* Q *should exist, satisfying*
$E_Q[(S_{T_j} - K)^+ | F_0] B_j = C(K, T_0, T_j)$

It is important to point out that we do not assume that this measure is unique or that it will remain constant over time. In particular, we allow for the possibility that the measure might change stochastically.

Condition 5 (PRICE INFORMATION) *The prices are known today of plain-vanilla calls and puts for an arbitrarily large, but finite, number of expiries, and for a continuum of strikes. They are denoted by* $Call_0(S_0, K, T_j)$*,* $Put_0(S_0, K, T_j)$ *with* $j = 1, 2, ..., N$*.*

To lighten notation, the dependence of the maturities on the indices is omitted in the following. In practice the market only provides a finite (and rather small) number of actual price quotes for calls and puts for different strikes and maturities. It is assumed that a sufficiently smooth interpolation/extrapolation between and beyond these points has been adopted by the trader, so that the resulting smile surface (to be defined precisely below) should give rise to twice-strike-differentiable implied unconditional densities. In practice, this point is not trivial, but a vast body of literature exists [References] and therefore we do not pursue this line of enquiry here.

Definition 1 (ADMISSIBLE SMILE SURFACE) *Define a smile surface such that the associated call prices satisfy*

$$\frac{\partial Call(t, T, S_t, T)}{\partial K} < 0 \tag{1}$$

$$\frac{\partial^2 Call(t, T, S_t, T)}{\partial K^2} > 0 \tag{2}$$

$$\frac{\partial Call(t, T, S_t, T)}{\partial T} > 0 \tag{3}$$

$$\frac{\partial Put(t,T,S_0,T)}{\partial K} > 0 \tag{4}$$

$$Call(t,T,S_t,T)|_{K=0} = S_t \tag{5}$$

$$lim_{K\to\infty}Call(t,T,S_t,K) = 0 \tag{6}$$

*an **admissible smile surface**.*

Condition 6 (ADMISSIBILITY) *We shall always assume that today's smile surface is admissible.*

Admissible smile surfaces prevent the possibility that, for instance, more out-of-the-money calls should be worth more than more-in-the-money calls; or require a strictly positive price density. For today's smile surface, the admissibility conditions are necessary and sufficient in order to rule out the possibility of static strategies *constructed today* that can be arbitraged[3]. When the smile surface in question is in the future, however, we shall show that the admissibility conditions are necessary, but not sufficient for absence of model-independent arbitrage.

2.2 The Building Blocks

Definition 2 (CURRENT SMILE SURFACE) *Let the implied volatility today, $\sigma_{impl}(t_0, T, K, S_0)$, be the function of strike and maturity that produces the number which, input in the Black and Scholes formula, $BS()$, for any strike and expiry and with today's value of the underlying, gives today's market price for the corresponding call:*

$$\sigma_{impl}(0) = \sigma_{impl}(t_0, T, K, S_0) : Call_0(S_0, K, T) =$$

$$= BS(S_0, K, T, \sigma_{impl}(t_0, T, K, S_0)) \qquad \forall K, T \tag{7}$$

*The quantiy $\sigma_{impl}(t_0, T, K, S_0)$as a function of T and K is also referred to as the **current smile surface**.*

Remark 1 *In the following the derivatives of the $BS()$ function with respect to the strike will be required. Unlike the derivatives of the function $BS()$ with respect to the underlying, that require information about how the smile surface changes when the stock price changes, and therefore depend on the true process of the underlying, the derivatives with respect to the strikes do not require such knowledge and can be simply observed from the (smoothly interpolated) market prices today (by our assumption, the call prices are available for a continuum of strikes). These derivatives are therefore independent of the true process.*

[3]An example of such a static strategy, for instance, would be the purchase of a call of strike K_1, and the sale for a higher price of a same-maturity call of strike K_2, with $K_2 > K_1$. The strategy is static because it does not require re-adjustment until expiry of both options.

Remark 2 *One could write* $\sigma_{impl} = \sigma_{impl}(t, T, K, S)$*, concentrate on the dependence of the implied volatility function on* S*, posit a process for* S *(e.g. a stochastic-volatility diffusion) and derive using Ito's lemma the dynamics of the implied volatility function. This is the route followed by Schoenbucher (1998). Since we want to avoid speaking about the process for the stock, and we do not want to be constrained by the choice of a particular process, we do not pursue this route.*

Remark 3 *The use of the implied volatility quantity in conjunction with the Black and Scholes formula should not be taken as a market endorsement of the process posited by the Black and Scholes model. Rather, an implied volatility is a conventional, market-agreed-upon way of quoting a price; it is simply 'the wrong number to put in the wrong formula to get the right price' (Rebonato (1999)).*

Definition 3 (STATE OF THE WORLD) *Let a* ***state of the world at time*** t*,* X_t*, be described by the value of the underlying plus the associated values of calls for all possible strikes and expiries:* $X_t = \{S_t \cup Call_t(S_t, K, t, T), \forall K, T\}$

Remark 4 *The value of the corresponding put is derived from call/put parity. Puts are therefore not explicitly analyzed in the following, but a symmetric treatment in terms of puts rather than calls could be presented.*

Definition 4 (FUTURE CONDITIONAL SMILE SURFACE) *Given a state of the world* X_t*, the* ***future (time-t) conditional implied volatility,*** $\sigma_{impl}(t, T, K, X_t)$ *is the function of strike and maturity that, for any* $Call_t \in X_t$*, and for any strike and expiry, produces the number which, input in the Black and Scholes formula,* $BS()$*, with the associated value* S_t *of the underlying, gives the prices for the corresponding call in the state of the world* X_t*:*

$$\begin{aligned}\sigma_{impl}(t) &= \sigma_{impl}(t, T, K, X_t) : Call_t(S_t, K, t, T) = \\ &= BS(S_t, K, T, \sigma_{impl}(t, T, K, S_t)) \qquad \forall K, T \end{aligned} \tag{8}$$

The quantity $\sigma_{impl}(t, T, K, X_t)$ *as a function of* T *and* K *is also referred to as the* ***future (time-t) conditional smile surface****.*

Remark 5 *There is a one-to-one correspondence between a future call price for a given strike and maturity and its associated implied volatility,* $\sigma_{impl}(t, T, K, S_t)$*. Therefore, the state of the world at time t can be equivalently described by specifying the value at time t of the underlying and the associated future (time-t) conditional implied volatility surface.*

Remark 6 *If one knew what the true process for the underlying is, requiring that the future conditional smile surface should depend on the underlying and on all the call prices could be replaced by the condition that the implied volatility surface should depend on S_t, on the type of process, and of whatever additional stochastic processes (if any) (e.g. stochastic volatilities) define the process for the underlying. Definition [FUTURE CONDITIONAL SMILE SURFACE], is however, more general, because it does not require knowledge of the true process and of its parameters.*

3 Deterministic Smile Surfaces

3.1 Equivalent Descriptions of a State of the World

The definitions presented so far have been expressed in terms of smile surfaces. For many applications it is useful to establish a correspondence between smile surface and state (or price) densities. This can be accomplished by recalling the following well-known result: the current risk-neutral price density for time T, $f_0(S_T)$, i.e. the risk-neutral (unconditional) probability density that the underlying will have value S_T at time T, given that state X_0 prevails today, is given by (see, e.g., Breeden and Litzenberger (1978) or Dupire (1994))

$$f_0(S_T) = \frac{\partial^2 Call_0(S_0, K, t_0, T)}{\partial K^2} = \frac{\partial^2 BS(S_0, K, t_0, T, \sigma_{impl}(t, T, K, S_0))}{\partial K^2} \quad (9)$$

As Breeden and Litzenberger point out, no assumptions have to be made regarding the stochastic process for the underlying in order to arrive at Equation 9, and individual preferences and beliefs are not restricted in any way.

Remark 7 *Even if all the unconditional densities obtained using Equation 9 were known exactly (for a continuum of times T) this would not determine uniquely the underlying process, which would only be unambiguously specified if all the* ***conditional*** *densities were provided as well. See, e.g., Baxter and Rennie (1998).*

Definition 5 (FUTURE CONDITIONAL RISK-NEUTRAL PRICE DENSITY) *The* ***future (time-t) conditional risk-neutral price density****, $f_t(S_T|X_t)$, is defined to be*

$$\begin{aligned} f_t(S_T|X_t) \quad = \quad & \frac{\partial^2 Call_t(S_t, K, t, T; X_t)}{\partial K^2} = \\ & \frac{\partial^2 BS(S_t, K, t, T, \sigma_{impl}(t, T, K, X_t))}{\partial K^2} \end{aligned} \quad (10)$$

i.e. it is the risk-neutral probability density that the underlying will have value S_T at time T given that state X_t prevails at time $t(T > t)$.

Definition 6 (FUTURE CONDITIONAL RISK-NEUTRAL STATE DENSITY) *The* ***future (time−t) conditional risk-neutral state density,*** $\Phi_t(X_T|X_t)$, *is defined to be the risk-neutral probability density that the world will be in state X_T at time T given that state X_t prevails at time t.*

Remark 8 *The state of the world at time t can be equivalently described in terms of*

i) the value at time t of the underlying plus the values of all the calls:

$$X_t = \{S(t) \cup Call_t(S_t, K, t, T), \forall K, T\} \tag{11}$$

ii) the value at time t of the underlying and the associated future (time-t) S_t-conditional implied volatility surface:

$$X_t = \{S_t \cup \sigma_{impl}(t, T, K, S_t), \forall K, T\} \tag{12}$$

iii) the value at time t of the underlying and the associated future (time-t) S_t-conditional risk-neutral density:

$$X_t = \{S_t \cup f_t(S_T|S_t)\} \qquad \forall T \tag{13}$$

From the properties of conditional expectations the following relationship must hold between present and future risk-neutral state densitites:

$$\Phi_0(X_{T_2}) = \int \Phi_0(X_{T_1})\Phi_{T_1}(X_{T_2}|X_{T_1})dX_{T_1} \tag{14}$$

Equation 14, which poses restrictions on the future conditional state densities, is very general, but not very easy to use in practice, since it conditions the expectation on the full state of the world (i.e. on a future realization of the stock price and of *all* the associated calls). It would be helpful to express condition 14 in a more manageable form, by conditioning only on the realization of the stock price. We explore below under what circumstances this is possible and meaningful - this will lead us directly to the concept of floating smile.

Definition 7 (DETERMINISTIC SMILE SURFACE) *Given an admissible smile surface today (time t_0) a* ***future, time-t conditional smile surface*** *is said to be* ***deterministic*** *if the future smile surface can be expressed at time t_0 as a deterministic function of time, maturity, strike and of the realization of the stock price at time t, S_t.*

This need not, in general be the case: stochastic volatility processes, for instance, do not produce future deterministic smile surfaces. Examples of processes that *do* generate deterministic smile surfaces are: the future smile surface produced by geometric-diffusion (Black-and-Scholes) process with constant and time dependent volatilities; by jump diffusion with constant or time-dependent coefficients; by displaced diffusions (Rubinstein (1983)) and their generalizations such as displaced jump-diffusions; by the Derman-Kani restricted-stochastic (local) volatility model; by the Variance Gamma process; etc.

3.2 Consequences of Deterministic Smile Surfaces

If the future smile surface is deterministic, given the knowledge of the future value of S_t the prices of all future calls are also known. Therefore, the state X_t is fully determined by S_t, and the (conditional and unconditional) price and state densities coincide:

$$\Phi(X_t) = f(S_t) \tag{15}$$

$$\Phi(X_T|X_t) = f(S_T|S_t) \tag{16}$$

Deterministic smile surfaces are important because they allow us to work with the much simpler *price* densities rather than the state densitites. For any current price density, $f_0(S_0)$, and conditional deterministic price density, $f_t(S_T|S_t)$, it is always true that

$$f_0(S_{T_2}) = \int f_0(S_{T_1}) f_{T_1}(S_{T_2}|S_{T_1}) dS_{T_1} \tag{17}$$

Using Equation 9 one can therefore write:

$$f_0(S_{T_2}) = \frac{\partial^2 BS(S_0, K, t_0, T_2, \sigma(t_0, T_2, S_0, K))}{\partial K^2}|_{K=S_{T_2}} \tag{18}$$

$$f_0(S_{T_1}) = \frac{\partial^2 BS(S_0, K, t_0, T_1, \sigma(t_0, T_1, S_0, K))}{\partial K^2}|_{K=S_{T_1}} \tag{19}$$

$$f_{T_1}(S_{T_2}|S_{T_1}) = \frac{\partial^2 BS(S_t, K, T_1, T_2, \sigma(T_1, T_2, S_{T_1}, K))}{\partial K^2}|_{K=S_{T_1}} \tag{20}$$

Note carefully that Equations 18 and 19 can be evaluated given today's call prices for the two maturities T_1 and T_2. The associated densities are therefore market-geiven. The same does not apply to the conditional density in Equation 20.

3.3 Kolmogorov-Compatible Deterministic Smile Surfaces

In order to lighten notation, denote the operator

$$\frac{\partial^2 BS}{\partial K^2}[\,] \equiv \Theta[\,] \tag{21}$$

Then Equation 20 can be re-written as

$$\Theta(S_0, K, t_0, T_2, \sigma(t_0, T_2, S_0, K)) = \tag{22}$$

$$\int \Theta(S_0, K, t_0, T_1, \sigma(t_0, T_1, S_0, K))\Theta(S_t, K, T_1, T_2, \sigma(T_1, T_2, S_{T_1}, K))dS_{T_1}$$

with $\Theta(S_0, K, t_0, T_1, \sigma(t_0, T_1, S_0, K))$ and $\Theta(S_0, K, t_0, T_2, \sigma(t_0, T_2, S_0, K))$ market-given, and $\Theta(S_t, K, T_1, T_2, \sigma(T_1, T_2, S_{T_1}, K))$ is to be determined so as to satisfy Equation 22.

Remark 9 *There is a one-to-one correspondence between the quantity*

$$\Theta(S_t, K, T_1, T_2, \sigma(T_1, T_2, S_{T_1}, K))$$

and future conditional deterministic densities (conditional future smile surfaces). There is in general an infinity of solutions

$$\Theta(S_t, K, T_1, T_2, \sigma(T_1, T_2, S_{T_1}, K))$$

such that Equation 22 is satisfied. Therefore, even if we require the smile surface to be deterministic, there still exists an infinity of future smile surfaces compatible with today's prices of calls and puts.

Definition 8 (KOLMOGOROV COMPATIBILITY) *Define any future deterministic conditional density or smile surface such that Equation 22 is satisfied a **Kolmogorov-compatible density**.*

Proposition 1 *Given a current admissible smile surface, if all the future deterministic smile surfaces for times $T_1, T_2, ..., T_n$ are Kolmogorov compatible no model-independent strategy revised on the same set of dates can generate arbitrage profits.*

3.4 Conditions for Uniqueness of Kolmogorov-Compatible Densities

The equations obtained up to this point determine the links between the present and the future densities that must be satisfied by deterministic smile surfaces in order to avoid model-independent arbitrage. One extra condition is required in order to ensure uniqueness of the resulting conditional density. This condition is often implicitly assigned by popular process-based models, and is derived below.

Condition 7 (DISTANCE CONDITION) *Let us assume that a Kolmogorov-compatible conditional probability density is of the form*

$$f(S_T|S_t) = f'(P(S_T) - P(S_t)) \tag{23}$$

for some functions $f'()$ and $P()$. If this is the case, the probability density is said to satisfy the distance condition.

Equation 23 requires that the transition probability of the stock price at two different times should only depend on the distance between (some function of) the starting and arrival points (whence the name 'Distance Condition'). If the function $P(S) = S$ one recovers a normal diffusion for the underlying. If $P(S) = ln(S)$ one recovers a log-normal diffusion. If $P(S) = S + \alpha$ or $P(S) = \ln(S + \alpha)$ we are in a (normal or log-normal, respectively) displaced-diffusion case; etc.

Remark 10 *The current price density can always be written as some function f'_0 of $P(S_0)$. If the [DISTANCE CONDITION] is satisfied, the current risk-neutral density for the function P of the underlying for time T_2, f', can be written as a convolution:*

$$f'_0(P_{T_2}) = \int f'_0(P_{T_1}) f'(P_{T_2} - P_{T_1}) dP_{T_1} =$$
$$= f'_0(P_{T_1}) * f'_0(P_{T_2}) \tag{24}$$

where the symbol $*$ indicates convolution, and, to lighten notation, we have denoted by P_{T_1} the quantity $P(S_{T_1})$.

Proposition 2 *Denote by $\mathcal{F}$ and $\mathcal{F}^{-1}$ the Fourier and the inverse Fourier transform operators, respectively. Then, given times T_1 and T_2, if a soultion exists, there is a unique future deterministic time-T_1 conditional density (smile surface) for expiry at time T_2 compatible with today's state of the world/ (with todays' prices for plain-vailla calls and puts)/(with today's smile surface). It is given by*

$$f'_{T_1}(P_{T_2} - P_{T_1}) = \mathcal{F}^{-1}[\frac{\mathcal{F}[f'_0(P_{T_2})]}{\mathcal{F}[f''_0(P_{T_1})]}] \tag{25}$$

Therefore, under the [DISTANCE CONDITION], the future, conditional risk-neutral density, and, therefore, the future conditional smile surface can be obtained from the market-given risk neutral densities. More precisely, if the [DISTANCE CONDITION] is satisfied, for a given P, if a solution exists it is unique, i.e. there exists a unique Kolmogorov-compatible future density.

3.5 Floating Smiles

One of the potentially desirable conditions for a smile function is that it should be self-similar when its arguments S_t and t undergo certain transformations. In particular, we can ask the following questions:

- What will the smile surface look like when the underlying changes?
- What will the smile surface look like when we move forward in time?

The answer to the the first question leads to the concept of floating or sticky smiles. The second question is related to the existence or otherwise of an arbitrage-free forward-propagated smile. To make these concepts precise in the deterministic case first we procede as follows.

The smile surface today, i.e. for a fixed S_0, can always be written as a function, $\widehat{\sigma}()$, of $ln[K/S_0] \equiv y_0$:

$$\sigma_{impl}(t_0, T, K, S_0) = \widehat{\sigma}(t_0, T, ln[K/S_0]) = \widehat{\sigma}(t_0, T, y_0) \tag{26}$$

This observation is useful in establishing the following conditions, which are central to the treatment to follow.

Condition 8 (STOCK HOMOGENEITY) *Let us impose that the time-t smile surface is deterministic and that it should be of the form*

$$\sigma_{impl}(t, T, K, S_t) = \widehat{\sigma}(t, T, y_t) \tag{27}$$

with $y_t \equiv ln[\frac{K}{S_t}]$ *and* $\widehat{\sigma}$ *the same function that describes the current smile surface.*

Definition 9 (FLOATING SMILE) *A future deterministic smile surface such that Condition [STOCK HOMOGENEITY] is satisfied for all t is called a* ***floating smile surface****.*

Since the definition of deterministic smile surface requires that Condition [STOCK HOMOGENEITY] should hold for any t, it must be true also for an instantaneous change in the stock price. This condition therefore directly relates to the translation properties (in log space) of the smile surface with the stock price. However, we do not know yet whether, and under which conditions, such a floating surface can exist without allowing arbitrage. This would certainly be the case if there were a process that produces deterministic floating smile surfaces, but we have not based our treatment on the specification of a particular process, and we must, therefore, follow some other route. In our language, the condition necessary for the existence of a deterministic floating smile is the following:

Proposition 3 *Let us assume that Conditions [STOCK HOMOGENEITY] and [DISTANCE CONDITION] are satisfied. If the conditional probability density is of the form $f(S_T|S_t) = \xi(lnS_T - lnS_t)$, i.e. if the function $P()$ in remark 21 and Proposition 1 is given by $P \equiv ln(S)$ (and f' is therefore the probability density for $ln[S_t]$), then the corresponding future smile surface is floating.*

Remark 11 *All pairs $\{K, S_t\}$ such that their ratio is a constant (and for fixed t and T) produce the same value for y_t. Therefore, if the smile surface is floating, all such pairs $\{K, S_0\}$ give rise to the same implied volatility, and a set of call prices simply proportional to S_t (given the homogeneity properties of the Black and Scholes formula).*

Definition 10 (FORWARD-PROPAGATED SMILE) *A floating smile surface such that the function that gives the implied volatility as a function of residual maturity and strike is independent of calendar time is said to be forward-propagated*

Remark 12 *There is no guarantee, in general, that a forward-propagated smile will be Kolmogorov-compatible (ie, that today's prices admit a forward-propagated smile without allowing model-independent arbitrage opportunities). If the trader felt that forward-propagation were a desirable property, she could try to find the future conditional densities (smile surfaces) that are Kolomogorov-compatible, and that are 'closest' - given some suitably defined distance - to forward-propagated densities (smile surfaces)*

Summarizing:

- **Deterministic smile**: there exists some function of maturity T, strike K and time t such that, conditional on the future stock price being known, the future smile surface is known today exactly
- **Floating smile**: there exists some function today of time t, maturity T and of the ratio $y_t = \ln \frac{K}{S_t}$ in terms of which one can express today all the future smile surfaces
- **Forward-propagated smile**: there exists some function today of residual maturity T and of the ratio $y_t = \ln \frac{K}{S_t}$ in terms of which one can express today all the future smile surfaces

3.6 Stochastic Smiles

We intend to describe the stochastic evolution of smiles, by describing today's smile as a function of a number of parameters, $\{\alpha\}$, and by assigning a stochastic behaviour to these parameter. This is done as follows.

Definition 11 *Denote by* $\Im_\alpha$ *the natural filtration generated by the stochastic evolution of the random variables* $\{\alpha\}$*. Denote by* $\Im_S$ *the natural filtration generated by the stochastic evolution of the random variable S. Denote by* $\Im_{S,\alpha}$ *the natural filtration generated by the stochastic evolution of the random variables S and* $\{\alpha\}$*. We assume that we are given a probability space* $(\Omega, P, \Im_{S,\alpha})$ *which satisfies the 'usual conditions'.*

Condition 9 (STOCHASTICITY) *Let us impose that the future (time-t) conditional implied volatility function,* $\sigma_{impl}(t, T, K, S_t)$*, should be a stochastic quantity, whose values depend on the realization of a discrete set of random variables* $\{\alpha_t\}$*.*

Condition 10 (DISCRETENESS) *Let us assume that the random variables* $\{\alpha\}$ *that determine the realization of the future implied volatility surface (the future conditional density) can assume an arbitrary large but finite number of values. Let* $\{\pi_{ij}^t\}$ *denote the probability of i-th realization of the j-th parameter*

α *at time t.*

Remark 13 *The approach is superficially similar to that employed by Joshi and Rebonato (2001) in their stochastic-volatility extension of the LIBOR market model. The important difference is that we assume in this work that the parameters describing the* ***implied*** *volatility surface are stochastic. Joshi and Rebonato (2001), on the other hand, assume that the parameters of the* ***instantaneous*** *volatility are stochastic. This apparently minor difference ensures automatically that all the resulting future smile surfaces are Kolmogorov-compatible, and arbitrage free.*

Condition 11 (INDEPENDENCE) *Let us assume that future smile surface can be written as* $\sigma_{impl}^t = \sigma(t, T, y_t; \{\alpha_t\})$ *and that the values of these random variables* α *at time t should be independent of* $y(t)$*:*

$$Prob(\alpha_t | y_t) = Prob(\alpha_t) \rightarrow Prob(\alpha_t, y_t) = Prob(\alpha_t)Prob(y_t) \tag{28}$$

As for Condition [STOCHASTICITY], the random variables $\{\alpha\}$ could be of very different nature: they could, for instance, be the second, third, fourth, etc. moments of a future probability density; they could be the future market prices of at-the-money volatilities, straddles and risk reversals; they could be the coefficients of a parametrically fitted density (see, e.g. Mirfendereski and Rebonato (2001), Samuel (2002)). All these interpretations are possible, as long as the random variables, however chosen, are independent of y_t. Whether, and to what extent, it is realistic and appropriate to make, say, the third moment or the skew independent of the underlying is clearly a modelling/empirical question (see the discussion in Section 4). Similar assumptions are however also embedded, often in a less transaprent way, in process-based modelling approaches, and our conditions at least force the trader to specify clearly her modelling assumptions in the 'language' that she prefers.

3.7 Stochastic Floating Smiles

Definition 12 (STOCHASTIC FLOATING SMILE) *Let us assume that a smile surface satisfies conditions [STOCHASTICITY] and [INDEPENDENCE], and let $\varphi(\alpha_1^t, \alpha_2^t, .., \alpha_n^t)$ denote the joint probability of occurrence of $\{\alpha_1^t, \alpha_2^t, .., \alpha_n^t\}$ at time t. Then, if it is possible to choose a measure Q, such that the time-τ expectation of $\widehat{\sigma}$ over the stochastic variables $\{\alpha\}$ at time t ($t > \tau$)is equal to its time-τ value*

$$
\begin{aligned}
E_Q[\widehat{\sigma}_{\alpha_t}(t,T,y_t)] &= \\
\int_{\alpha_t} \widehat{\sigma}(t,T,y_t)\varphi(\alpha_1,\alpha_2,..,\alpha_n)d\alpha_1 d\alpha_2,..,d\alpha_n &= \widehat{\sigma}_{\alpha_0}(\tau,T,y_\tau)
\end{aligned}
\tag{29}
$$

then a future implied volatility function generated by the processes $\{\alpha\}$ is said to produce a ***floating stochastic volatility surface***[4]*.*

3.8 Introducing Equivalent Deterministic Smile Surfaces

Remark 14 *If no arbitrage is to be allowed, a probability measure must exist such that the relative price of a call today is given by the weighted expectation of the relative call price (payoff) at time t. Let us assume that we are dealing with a stochatsic smile surface such that conditions [INDEPENDENCE] and [STOCHASTICITY] are satisfied. Then, if the numeraire is chosen to be the (deterministic) discount bond maturing at time T, $Z(0,T)$, one can write, for $t \leq T$,*

$$\frac{Call_0(S_0,T)}{Z(0,t)} = Z(t,T)E_P[Call_t|\Im_{S,\alpha_0}] =$$

$$Z(t,T)\sum \pi_i \int Call_t(S_t,T,X_i(t))f_0(S_t)dS_t =$$

$$= Z(t,T)\sum \pi_i \int BS(S_t,t,T,\sigma_{impl}(t,T,S_t,K;X_i(t))f_0(S_t)dS_t =$$

$$= Z(t,T)\sum \pi_i \int BS(S_t,t,T,\widehat{\sigma}_{\alpha_i}(t,T,y_t))f_0(S_t)dS_t =$$

[4]Traders tend to use the terms 'floating' and 'sticky delta' interchangeably. Indeed, it is easy to show that the definition of floating smile we provide produces, in the special case of a deterministic future smile surface, the same value for the delta for any fixed value of y_t, and hence the 'sticky delta' condition. Therefore, our definition of floating smile is consistent with and generalizes the market intuition of a 'sticky-delta' smile.

$$= Z(t,T) \sum \pi_i \int [\int (S_T - K)^+ f_t(S_T|S_t)_i dS_T] f_0(S_t) dS_t \qquad (30)$$

where

π_i *is the probability of the i-th realization of the multiplet* $\{\alpha\}$.Note carefully that the quantity y_t depends on the strike, K.

Proposition 4 *If the smile is floating, the price of a call today in the presence of a stochastic floating smile is identical to the price that would obtain with the single deterministic stock-homogeneous (floating) future smile associated with the average conditional density* $\overline{f_t}(S_T|S_t)$. *Such a smile is called the* ***equivalent deterministic future smile****.*

Proof. From Equation 30, after interchanging the order of integration one obtains:

$$Call_0(S_0,T)/Z(0,T) =$$

$$= \sum \pi_i \int [\int (S_T - K)^+ f_t(S_T|S_t)_i dS_T] f_0(S_t) dS_t =$$

$$= \int [\sum \pi_i \int (S_T - K)^+ f_t(S_T|S_t)_i dS_T] f_0(S_t) dS_t =$$

$$= \int [\int (S_T - K)^+ \sum \pi_i f_t(S_T|S_t)_i dS_T] f_0(S_t) dS_t =$$

$$= \int [\int (S_T - K)^+ \overline{f_t}(S_T|S_t) dS_T] f_0(S_t) dS_t =$$

$$= \int BS(S_t, t, T, \overline{\sigma}(t, T, y_t)) f_0(S_t) dS_t \qquad (31)$$

where $\overline{f_t}(S_T|S_t)$ is the average conditional density, and $\overline{\sigma}(t,T,y_t)$ the associated deterministic floating smile (implied volatility). ■

For any stochastic future floating smile there always corresponds a deterministic floating future smile that produces identical prices for all calls and puts. This deterministic smile is the equivalent deterministic future smile.

3.9 Implications of the Existence of an Equivalent Deterministic Smile Surface

A variety of processes have been proposed in order to describe the stock price dynamics. Each of these processes gives rise to a set of future smile surfaces. In some cases these smile surfaces are deterministic, in other stochastic. A stochastic-smile-surface stock price process will produce prices for calls and puts different from the prices from the equivalent deterministic future smile only to the extent that it produces stochastic future smiles which are not independent of the future realization of the stock price. Therefore the ability to create a

future stochastic smile surface correlated with a future realization of the stock price is the only feature of a model that can provide anything more complex than a deterministic-smile setting.

Very often, the complexity of today's observed smile surface is taken as an indication that a model that produces stochastic future smiles must be employed in order to account for today's observed prices. This might well be the case, but only if the probability of occurence of future smiles is not independent of the future stock price realization.

3.10 Extension to Displaced Diffusions

Empirical observations indicate that there exists a negative correlation between the future level of smile surfaces and of stock prices. This would seem to invalidate one of the crucial conditions of the approach outlined above. As long as the dependence is relatively simple, however, the approach can sometimes be rescued by a simple change of variables. One possible way to do this is to recast the distance condition in terms of a function other than $\ln(S_t)$. Another attractive route is to employ the approximate but accurate equivalence between CEV process and displaced-diffusion processes (Marris, Rubinstein). This can be achieved as follows.

Definition 13 *Define the present or future a-displaced implied volatility,* σ^a_{impl}*, as the quantity that, input in the Black-and-Scholes formula with* $(S+a)$ *as spot and* $(K+a)$ *as strike produces the value of a call with spot equal to* S *and strike equal to* K*. Also, define*

$$y^a_t \equiv ln[\frac{K+a}{S_t+a}] \tag{32}$$

Mutatis mutandis, all the treatment presented above can be re-cast in terms of the new quantity y^a_t. Clearly, the distance condition is now expressed in terms of the function $\ln(S_t + a)$.

4 Empirical tests

We want to examine stochastic volatility models; jump-diffusion models; variance gamma; Johnson's model.

5 Limitations and suggestions for future work

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