Implementations of the LIBOR Market Model

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Outline

• The components of the model
• The continuous-time evolution
• Simulation: long-stepping and drift issues
• Methods in current use
  – Predictor-Corrector
  – Glasserman-Zhao
• Variants of predictor-corrector
  – Balland
  – Pietersz –Pelsser-Regenmortel
• New methods
• Numerical comparison of performance
Components of the Model

- Tenor Dates \(0 < T_0 < T_1 < \ldots < T_n\)
- Set \(\tau_i = T_{i+1} - T_i\)
- Forward Rates \(f_0, f_1, \ldots, f_{n-1}\)

- \(f_i(t)\) is the simple interest rate for borrowing from \(T_i\) to \(T_{i+1}\)
- i.e. at time \(t\) one can obtain a contract to receive £1 at time \(T_i\) and repay £\((1+f_i(t)\tau_i)\) at time \(T_{i+1}\).
The continuous-time evolution

The forward rates have a correlated log-normal evolution:

\[
\frac{df_i}{f_i} = \sigma_i(t)dW_i + \mu_idt
\]

- the volatility and correlation are deterministic functions of time.
The drift term

- We will take the zero-coupon bond maturing at time $T_n$ as numeraire.
- The choice of numeraire and no-arbitrage considerations enforce the drift term.
- This is the unique drift that makes the ratios of tradables to the numeraire martingales.

$$
\mu_i = - \sum_{k=i+1}^{n-1} \frac{\tau_k f_k}{1 + \tau_k f_k} \rho_{ik} \sigma_i \sigma_k
$$
Pricing in LMM

The price of a derivative, \( D \), is given by the expectation of its discounted pay-off. If it produces a cashflow at time \( T \), then

\[
\frac{D(0)}{Z(0)} = \mathbb{E}(\frac{D(T)}{Z(T)}),
\]

where the expectation is taken under the pricing measure, which depends on the numeraire \( Z \).

We typically evaluate this via Monte Carlo. To do so, we have to be able to sample from the distribution of the rates at a future time.
Long-stepping

Many non-callable pure LIBOR products have pay-offs that only depend on the LIBOR rates at their own reset times, e.g.,

- TARNs
- ratchets
- swap triggering on a FRA rate

• We refer to these as long-step products. They are the type Ia products in the Rebonato classification.
Terminal Distribution

- For long-step products, we only need to know the values $f_i(T_i)$, the rates at their own reset times.
- Equivalently, we think of the rates as having zero volatility after their own reset times, so we just need to know the values of $f_i(T_n)$ for each $i$ – the joint terminal distribution.
- Cash flows before time $T_n$ are rolled up to time $T_n$.
- We need only do one step in our Monte Carlo.
Pros of long-stepping

- Time reduction as fewer steps are required.
- Use of terminal bond as numeraire gives negative drift – rates unlikely to blow up.
- Lower effective dimensionality for multi-factor model. Faster Monte-Carlo convergence.
- No penalty from using full-factor models.
  - Terminal decorrelation means that the covariance matrix of all the rates to maturity is always full factor.
Cons of long-stepping

- No speed up is obtained from using low factor models
- Forced to use terminal bond as numeraire. The value of the numeraire can get very small if rates become large. In contrast the money-market account has value at least 1.
Stepping forward from S to T

We have

$$\log f_i(T) = \log f_i(S) + \int_S^T \sigma_i(t) dW_i - \frac{1}{2} \int_S^T \sigma_i^2(t) dt + \int_S^T \mu_i dt,$$

where

$$\mu_i = - \sum_{k=i+1}^{n-1} \frac{\tau_k f_k}{1 + \tau_k f_k} \rho_{ik} \sigma_i \sigma_k$$

The last integral is the difficult part – the drift term.
Drift issues

- Fundamental trickiness in implementing LMM is that drift is state-dependent.
- One therefore needs ways to approximate the evolution.
- Industry standard used to be Euler stepping using many steps.
- Now predictor-corrector or Glasserman-Zhao.
Euler stepping

Approximate the drift summand

\[
\int_S^T \frac{\tau_k f_k}{1 + \tau_k f_k} \rho_{ik} \sigma_i \sigma_k dt
\]

by

\[
\frac{\tau_k f_k(S)}{1 + \tau_k f_k(S)} \int_S^T \rho_{ik}(t) \sigma_i(t) \sigma_k(t) dt.
\]
Predictor-corrector

- Estimate rates using Euler stepping
- Re-compute state-dependent part at the end of each step
- Re-evolve rates using average drift with same random numbers.
- Iterative predictor-corrector – do one rate at a time working backwards
Equations for predictor-corrector

Obtain initial estimates, \( \tilde{f}_i(T) \), using Euler stepping.
Then recompute the drift term using these values and average:

\[
\frac{1}{2} \left( \frac{\tau_k f_k(S)}{1 + \tau_k f_k(S)} + \frac{\tau_k \tilde{f}_k(T)}{1 + \tau_k \tilde{f}_k(T)} \right) \int_S^T \rho_{ik}(t) \sigma_i(t) \sigma_k(t) \, dt.
\]

Noting that the drift \( \mu_i \) only involves \( f_k \) for \( k > i \), iterative predictor-corrector successively computes \( f_{n-1}(T) \), \( f_{n-2}(T) \), ... and uses these values in the above expression.
Glasserman-Zhao

- Evolve log bond-ratios instead of rates
- Drifts become zero.
- Volatilities become state-dependent.
- FRAs are priced right by construction.
- Internally, “arbitrage free”
- Many variants – e.g. second order schemes, log versus bond itself
Pietersz-Pelsser-Regenmortel

- Proceeds in the same basic manner as iterative predictor-corrector
- Having estimated $f_k(T)$, uses log-linear interpolation to estimate $f_k(t)$ at a few other points between $S$ and $T$
- Then uses numerical integration to estimate the drift integrals involving $f_k$
- Possibly also incorporates a volatility term to get the correct mean value of $f_k(t)$ conditional on $f_k(S)$ and $f_k(T)$
Balland

Uses geometric mean of rates to compute the drifts.

Set

$$\bar{f}_k = \sqrt{f_k(S) f_k(T)}$$

and approximate

$$\int_S^T \frac{\tau_k f_k}{1 + \tau_k f_k} \rho_{ik} \sigma_i \sigma_k dt \approx \frac{\tau_k \bar{f}_k}{1 + \tau_k \bar{f}_k} \int_S^T \rho_{ik}(t) \sigma_i(t) \sigma_k(t) dt.$$
Truncated Predictor-Corrector

- When long-stepping, predictor-corrector uses the estimated value of \( f_k(T_k) \) in calculating the drift of earlier rates \( f_i \) – despite the fact they reset at times \( T_i < T_k \).
- Suggests that interpolating the rate should give a better estimate – when calculating the drift of \( f_i \) use an estimate for \( f_k(T_i) \) instead.
- Derive these estimates in the same manner as PPR.
- Disappointing.
Correlation-adjusted predictor corrector (capc)

• Like truncated predictor-corrector, attempts to estimate the value of \( f_k(T_i) \) and use this in the drift of \( f_i \).

• Observe, for example, that \( f_{17}(T_5) \) may be much more strongly correlated with \( f_5(T_5) \) than it is with \( f_{17}(T_{17}) \).

• So, when calculating the drift of \( f_5 \), we obtain the best possible estimate for \( f_{17}(T_5) \) from the information \( \int_0^T \sigma_5 dW_5 f_{17}(T_{17}) \) and the volatility part of \( f_5(T_5) \) -
Correlation-adjusted numerical integration (cani)

- Very recent development
- In using capc when long-stepping, we make quick estimates of $f_k(T_i)$ for use in the drift calculation of $f_i$. ($i<k$).
- The drift term of $f_i$ is an integral from 0 to $T_i$. Use all the estimates $f_k(T_0), \ldots, f_k(T_i)$ and integrate numerically.
- Estimate only at a few points if this becomes too time-consuming.
Comparison of the different methods

We price various products and evaluate other quantities using just one long step.

When we know the correct value analytically we can measure the error precisely. Otherwise, we also price with successively shorter steps to see convergence to the correct value.

We just include the best versions of PPR, truncated predictor-corrector and Glasserman-Zhao.
In the following charts…

• we run enough paths so that Monte-Carlo error is not significant
• we typically use 20 annual rates: $T_0=1$ and $T_{20}=21$
• Initial rates are usually 5%
• Volatility is either flat 20% or 30% or of Rebonato-abcd form. We do not specify this precisely unless significant.
• Correlation is exponentially decaying. Rates $i$ and $j$ have instantaneous correlation

\[ e^{-\beta|T_i - T_j|} \]

Typically we take $\beta=10\%$ or $\beta=4\%$. 
Mean of 10th Forward rate

Length of each step

- pc (HJJ)
- Iterative pc
- Correlation-Adjusted pc
- Balland
- Truncated pc
- Glasserman-Zhao
- PPR
At-the-money FRAs - humped vol, high corr.
Pricing errors for at-the-money caplets

Index number of caplet

Error in price

-0.0006
-0.0004
-0.0002
0
0.0002
0.0004
0.0006

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20

HJJ
truncated-pc
Glasserman-Zhao
iterative pc
capc
Balland
PPR
out-of-the-money caplet pricing errors

index number of forward

pricing error in corresponding caplet

-0.0006
-0.0004
-0.0002
0
0.0002
0.0004

HJJ
truncated-pc
Glasserman-Zhao
iterative pc
capc
Balland
PPR
out-of-the-money caplet pricing errors

index number of forward

pricing error in corresponding caplet

-0.0006
-0.0004
-0.0002
0
0.0002
0.0004
0.0006

HJJ
truncated-pc
Glasserman-Zhao
iterative pc
capc
Balland
PPR
cani
out-of-the-money caplets - humped vol.

Graph showing the pricing error of corresponding caplets for different models.

- truncated-pc
- Glasserman-Zhao
- ipc
- capc
- Balland
- PPR

Index number of forward:

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20

Pricing error of corresponding caplet:

-0.0003 -0.0002 -0.0001 0 0.0001 0.0002 0.0003
A double digital. Pays 1 if the sixth rate exceeds 5% and the last rate exceeds 10%.
Up-and-in Trigger swap. Strike 5%, Trigger 8%. Humped vol.
Up-and-in Trigger Swap. Strike 5%
Triggers slide from 8.8% to 5%
Down and out trigger swap. Strike 5%.
Sliding triggers 3.1% to 5%
Down and out trigger swap. Strike 5%.
Sliding triggers 3.1% to 5%
Autocap. 7 annual caplets. Strike 5% Triggers 7%