RBS

The Royal Bank of Scotland Group

Implementations of the LIBOR Market Model

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Outline

- The components of the model
- The continuous-time evolution
- Simulation: long-stepping and drift issues
- Methods in current use
  - Predictor-Corrector
  - Glasserman-Zhao
- Variants of predictor-corrector
  - Balland
  - Pietersz –Pelsser-Regenmortel
- New methods
- Numerical comparison of performance

Components of the Model

- Tenor Dates $0<T_0<T_1<\ldots<T_n$
- Set $\tau_i=T_{i+1}-T_i$
- Forward Rates $f_{0,}f_1,\ldots,f_{n-1}$

  - $f_i(t)$ is the simple interest rate for borrowing from T_i to T_{i+1}
  - i.e. at time t one can obtain a contract to receive £1 at time T_i and repay £$(1+f_i(t)\tau_i)$ at time T_{i+1}.

The continuous-time evolution

The forward rates have a correlated log-normal evolution:

$$\frac{df_i}{f_i} = \sigma_i(t) dW_i + \mu_i dt$$

- the volatility and correlation are deterministic functions of time.

The drift term

- We will take the zero-coupon bond maturing at time T_n as numeraire.
- The choice of numeraire and no-arbitrage considerations enforce the drift term.
- This is the unique drift that makes the ratios of tradables to the numeraire martingales.

$$\mu_i = -\sum_{k=i+1}^{n-1} \frac{\tau_k f_k}{1+\tau_k f_k} \rho_{ik} \sigma_i \sigma_k$$

Pricing in LMM

The price of a derivative, D, is given by the expectation of its discounted pay-off. If it produces a cashflow at time T, then

$$D(0)/Z(0) = \mathbf{E}(D(T)/Z(T)),$$

where the expectation is taken under the pricing measure, which depends on the numeraire Z.

We typically evaluate this via Monte Carlo. To do so, we have to be able to sample from the distribution of the rates at a future time.

Long-stepping

Many non-callable pure LIBOR products have pay-offs that only depend on the LIBOR rates at their own reset times, e.g.,

- TARNs
- ratchets
- swap triggering on a FRA rate

- We refer to these as long-step products. They are the type Ia products in the Rebonato classification.

Terminal Distribution

- For long-step products, we only need to know the values $f_i(T_i)$, the rates at their own reset times.
- Equivalently, we think of the rates as having zero volatility after their own reset times, so we just need to know the values of $f_i(T_n)$ for each i – the joint terminal distribution.
- Cash flows before time T_n are rolled up to time $T_{n.}$
- We need only do one step in our Monte Carlo.

Pros of long-stepping

- Time reduction as fewer steps are required.
- Use of terminal bond as numeraire gives negative drift – rates unlikely to blow up.
- Lower effective dimensionality for multi-factor model. Faster Monte-Carlo convergence.
- No penalty from using full-factor models.
  - Terminal decorrelation means that the covariance matrix of all the rates to maturity is always full factor.

Cons of long-stepping

- No speed up is obtained from using low factor models
- Forced to use terminal bond as numeraire. The value of the numeraire can get very small if rates become large. In contrast the money-market account has value at least 1.

Stepping forward from S to T

We have

$$\log f_i(T) = \log f_i(S) + \int_S^T \sigma_i(t)\,dW_i - \frac{1}{2}\int_S^T \sigma_i^2(t)\,dt + \int_S^T \mu_i\,dt,$$

where

$$\mu_i = -\sum_{k=i+1}^{n-1} \frac{\tau_k f_k}{1+\tau_k f_k}\rho_{ik}\sigma_i\sigma_k$$

The last integral is the difficult part – the drift term.

Drift issues

- Fundamental trickiness in implementing LMM is that drift is state-dependent
- one therefore needs ways to approximate the evolution
- industry standard used to be Euler stepping using many steps
- Now predictor-corrector or Glasserman-Zhao

Euler stepping

Approximate the drift summand

$$\int_S^T \frac{\tau_k f_k}{1+\tau_k f_k} \rho_{ik} \sigma_i \sigma_k dt$$

by

$$\frac{\tau_k f_k(S)}{1+\tau_k f_k(S)} \int_S^T \rho_{ik}(t) \sigma_i(t) \sigma_k(t)\, dt.$$

Predictor-corrector

- Estimate rates using Euler stepping
- Re-compute state-dependent part at the end of each step
- Re-evolve rates using average drift with same random numbers.
- Hunter-Jäckel-Joshi *Risk* (2001)
- Iterative predictor-corrector – do one rate at a time working backwards

Equations for predictor-corrector

Obtain initial estimates, $\tilde{f}_i(T)$, using Euler stepping.
Then re-compute the drift term using these values and average:

$$\frac{1}{2}\left(\frac{\tau_k f_k(S)}{1+\tau_k f_k(S)} + \frac{\tau_k \tilde{f}_k(T)}{1+\tau_k \tilde{f}_k(T)}\right)\int_S^T \rho_{ik}(t)\sigma_i(t)\sigma_k(t)\,dt.$$

Noting that the drift μ_i only involves f_k for $k > i$, iterative predictor-corrector successively computes $f_{n-1}(T)$, $f_{n-2}(T), \ldots$ and uses these values in the above expression.

Glasserman-Zhao

- Evolve log bond-ratios instead of rates
- Drifts become zero.
- Volatilities become state-dependent.
- FRAs are priced right by construction.
- Internally, “arbitrage free”
- Many variants – e.g. second order schemes, log versus bond itself

Pietersz-Pelsser-Regenmortel

- Proceeds in the same basic manner as iterative predictor-corrector
- Having estimated $f_k(T)$, uses log-linear interpolation to estimate $f_k(t)$ at a few other points between S and T
- Then uses numerical integration to estimate the drift integrals involving f_k
- Possibly also incorporates a volatility term to get the correct mean value of $f_k(t)$ conditional on $f_k(S)$ and $f_k(T)$

Balland

Uses geometric mean of rates to compute the drifts.

Set

$$\bar{f}_k = \sqrt{f_k(S) f_k(T)}$$

and approximate

$$\int_S^T \frac{\tau_k f_k}{1+\tau_k f_k} \rho_{ik} \sigma_i \sigma_k dt \approx \frac{\tau_k \bar{f}_k}{1+\tau_k \bar{f}_k} \int_S^T \rho_{ik}(t) \sigma_i(t) \sigma_k(t) \, dt.$$

Truncated Predictor-Corrector

- When long-stepping, predictor-corrector uses the estimated value of $f_k(T_k)$ in calculating the drift of earlier rates f_i – despite the fact they reset at times $T_i < T_{k.}$
- Suggests that interpolating the rate should give a better estimate – when calculating the drift of f_i use an estimate for $f_k(T_i)$ instead.
- Derive these estimates in the same manner as PPR
- Disappointing.

Correlation-adjusted predictor corrector (capc)

- Like truncated predictor-corrector, attempts to estimate the value of $f_k(T_i)$ and use this in the drift of $f_{i.}$
- Observe, for example, that $f_{17}(T_5)$ may be much more strongly correlated with $f_5(T_5)$ than it is with $f_{17}(T_{17})$.
- So, when calculating the drift of f_5, we obtain the best possible estimate for $f_{17}(T_5)$ from the information con$\int_0^T \sigma_5 dW_5$ $f_{17}(T_{17})$ and the volatility part of $f_5(T_5)$ -

Correlation-adjusted numerical integration (cani)

- Very recent development
- In using capc when long-stepping, we make quick estimates of $f_k(T_i)$ for use in the drift calculation of f_i. ($i<k$).
- The drift term of f_i is an integral from 0 to T_i. Use all the estimates $f_k(T_0),\ldots,f_k(T_i)$ and integrate numerically.
- Estimate only at a few points if this becomes too time-consuming.

Comparison of the different methods

We price various products and evaluate other quantities using just one long step.

When we know the correct value analytically we can measure the error precisely. Otherwise, we also price with successively shorter steps to see convergence to the correct value

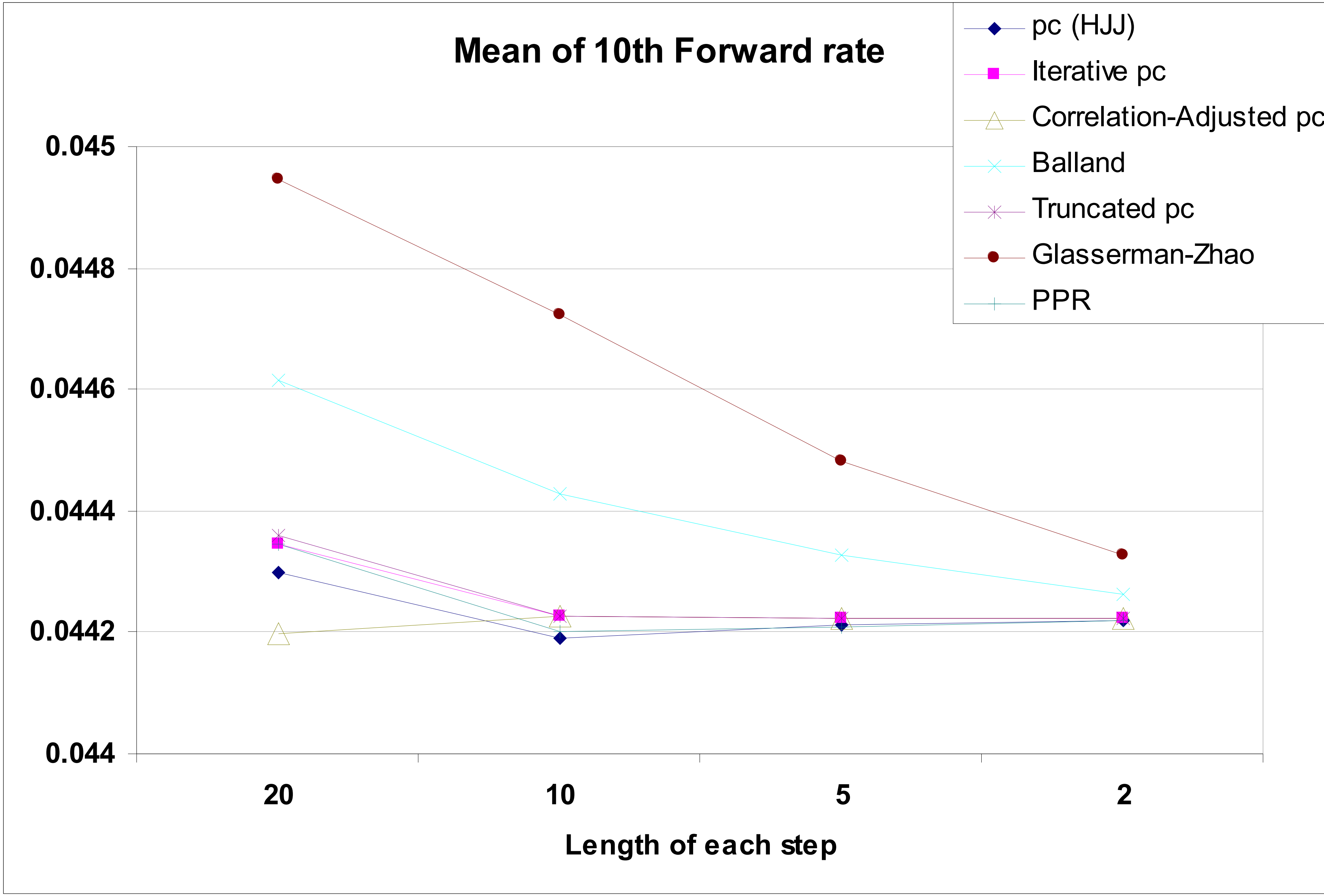
We just include the best versions of PPR, truncated predictor-corrector and Glasserman-Zhao

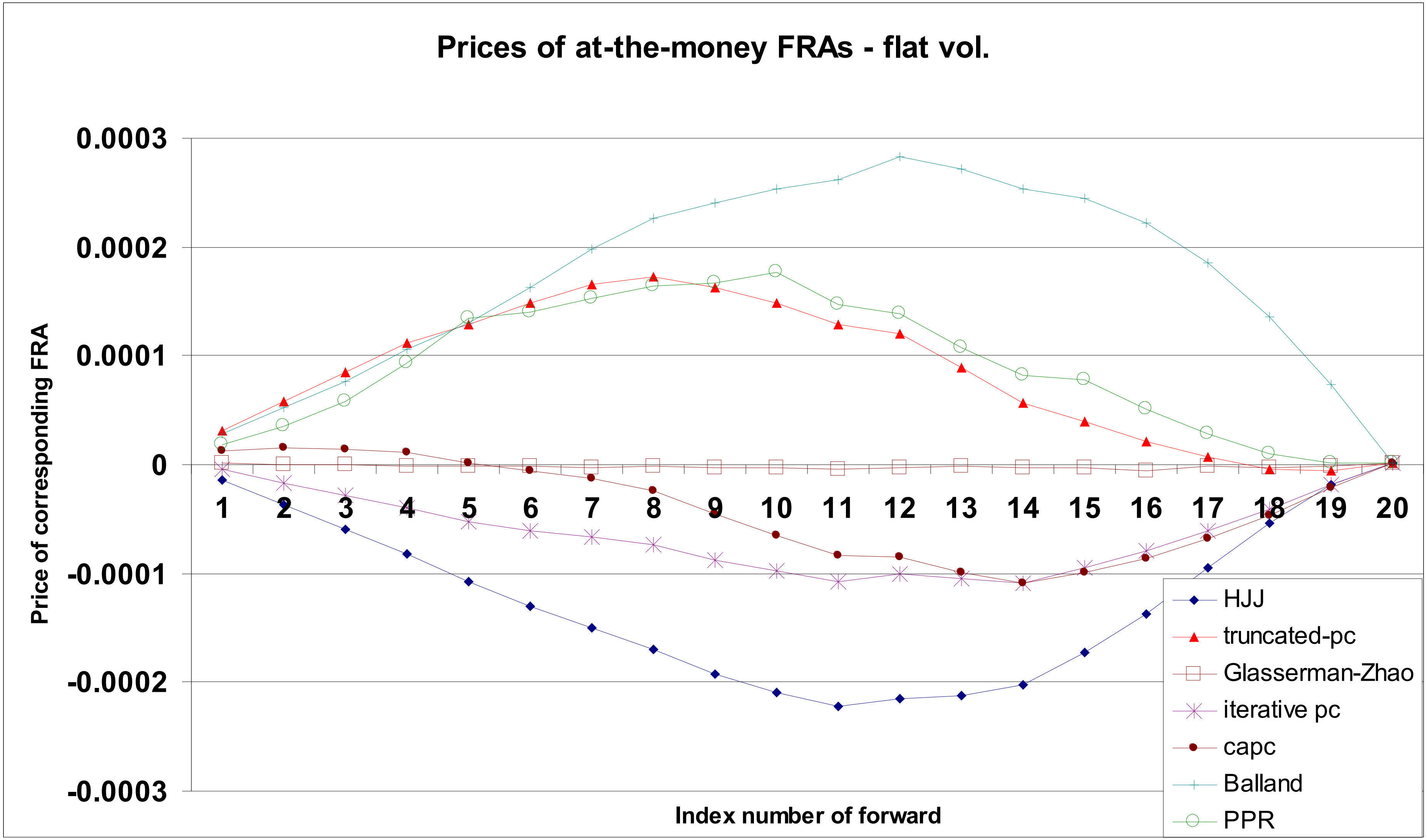
In the following charts…

- we run enough paths so that Monte-Carlo error is not significant
- we typically use 20 annual rates: $T_0=1$ and $T_{20}=21$
- Initial rates are usually 5%
- Volatility is either flat 20% or 30% or of Rebonato-abcd form. We do not specify this precisely unless significant.
- Correlation is exponentially decaying. Rates i and j have instantaneous correlation

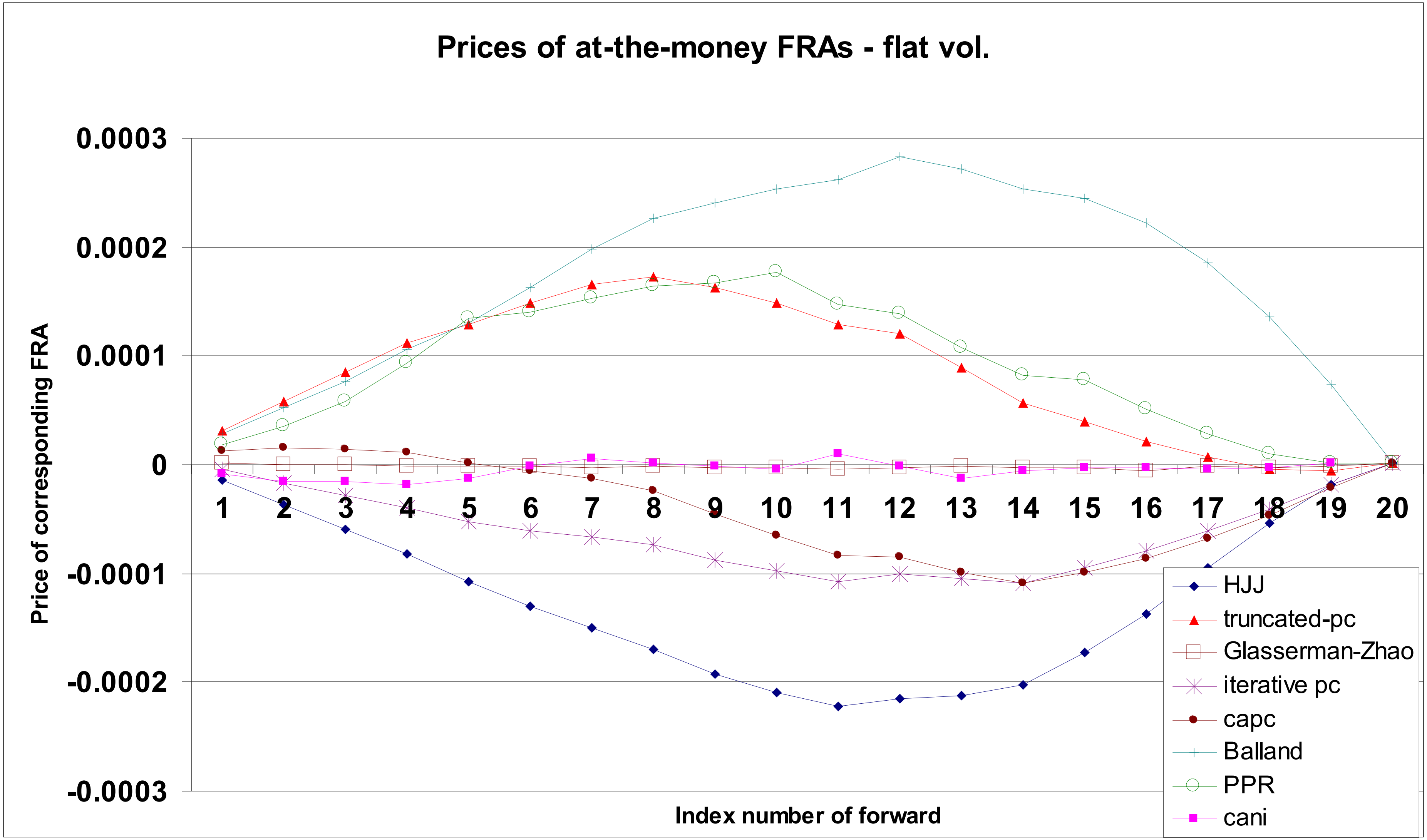
$$e^{-\beta|T_i-T_j|}$$

Typically we take β=10% or β=4%.

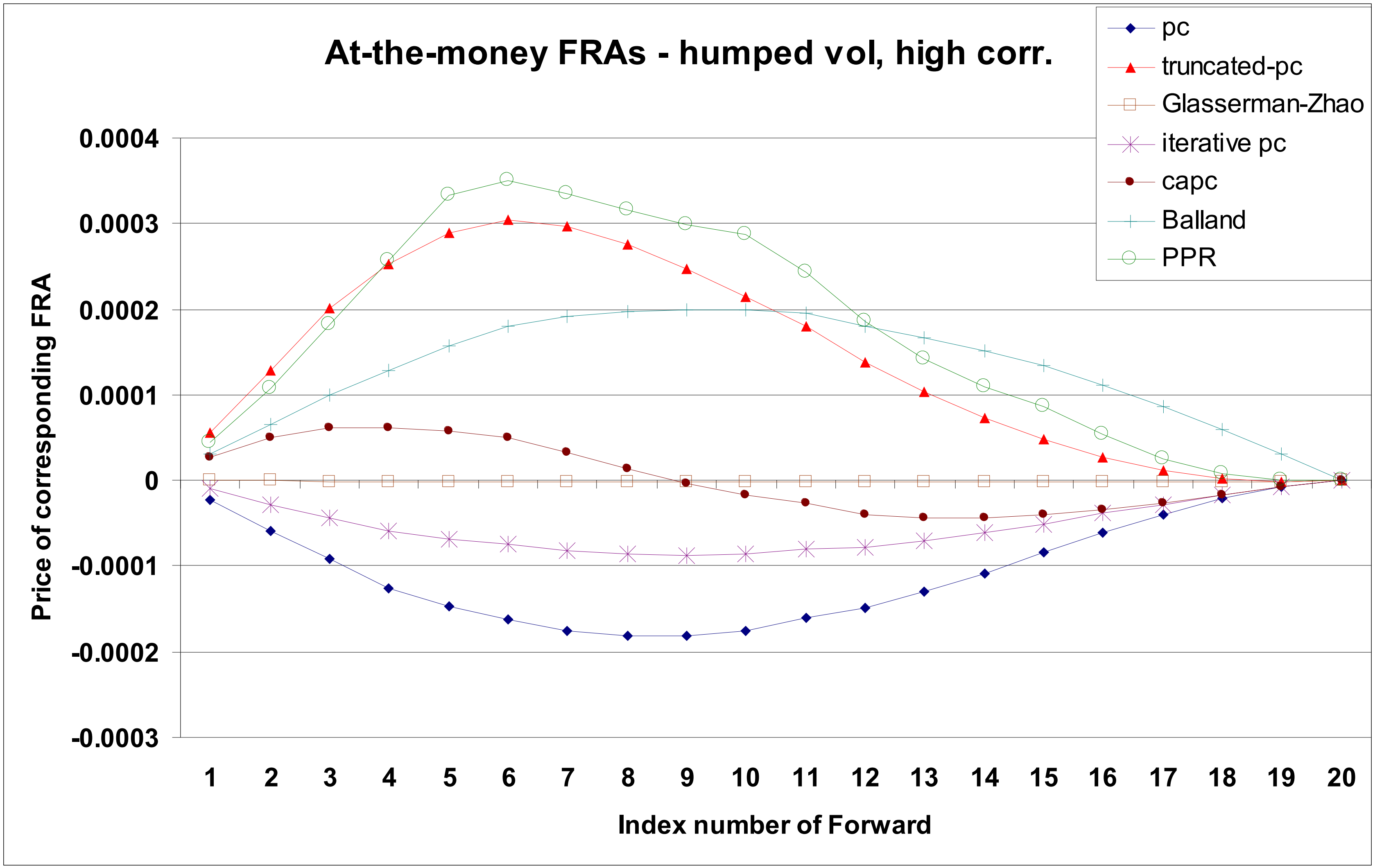

Mean of 10th Forward rate
pc (HJJ)
Iterative pc
Correlation-Adjusted pc
Balland
Truncated pc
Glasserman-Zhao
PPR
0.045
0.0448
0.0446
0.0444
0.0442
0.044
20
10
5
2
Length of each step

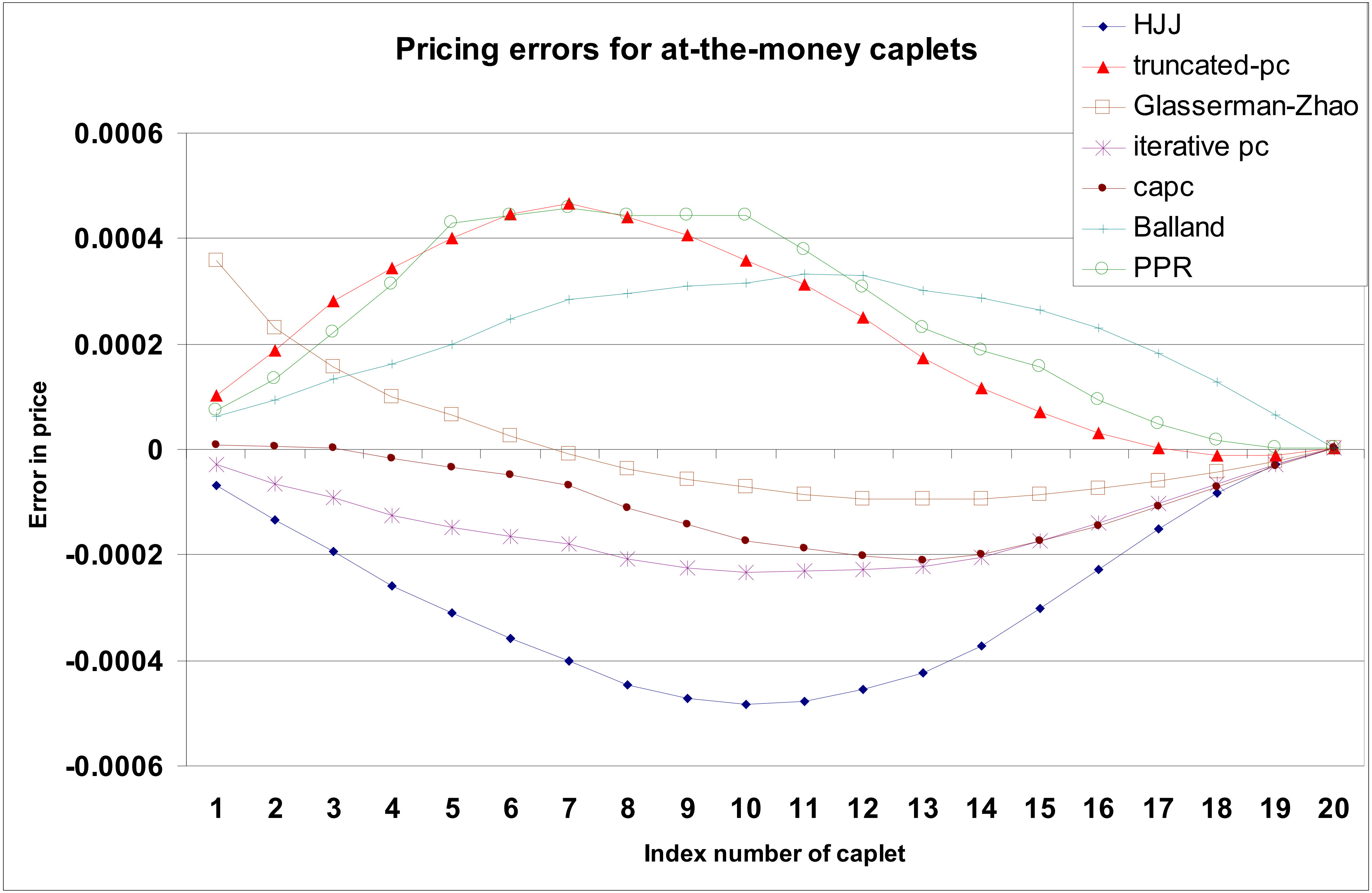


Prices of at-the-money FRAs - flat vol.
0.0003
0.0002
0.0001
0
-0.0001
-0.0002
-0.0003
Price of corresponding FRA
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20
Index number of forward
HJJ
truncated-pc
Glasserman-Zhao
iterative pc
capc
Balland
PPR

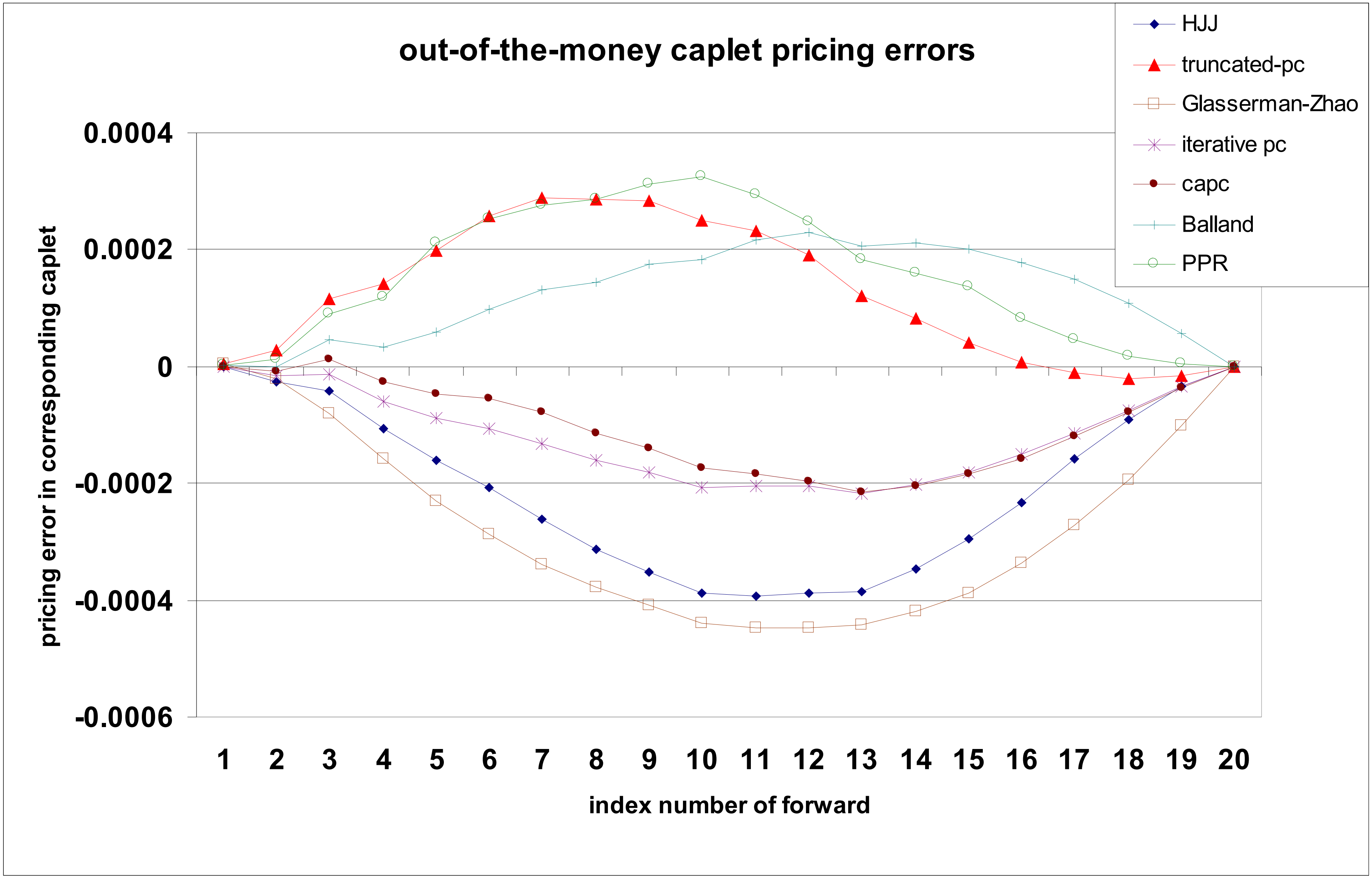


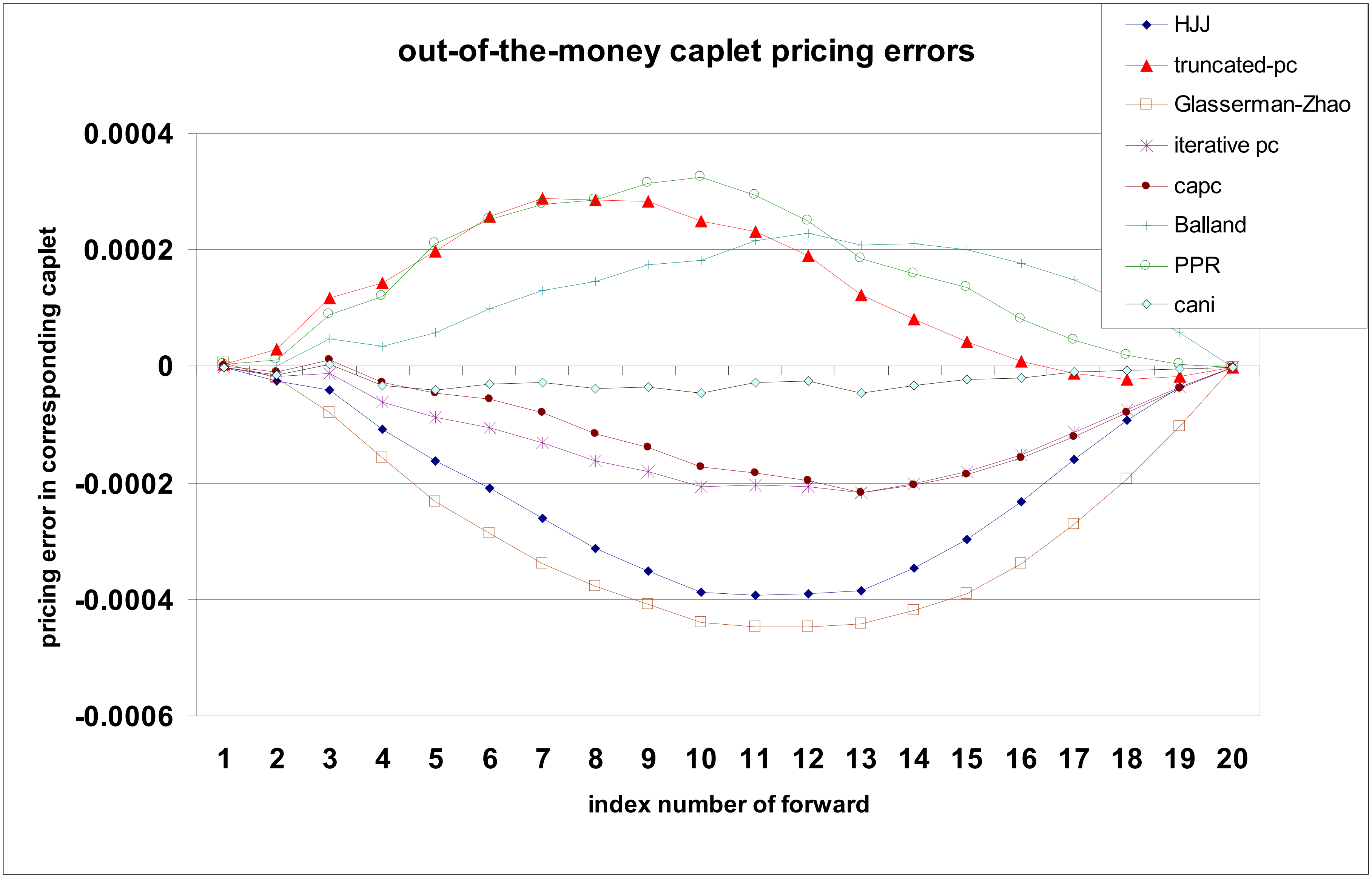
Prices of at-the-money FRAs - flat vol.
0.0003
0.0002
0.0001
0
-0.0001
-0.0002
-0.0003
Price of corresponding FRA
1
2
3
4
5
6
7
8
9
10
11
12
13
14
15
16
17
18
19
20
Index number of forward
HJJ
truncated-pc
Glasserman-Zhao
iterative pc
capc
Balland
PPR
cani


At-the-money FRAs - humped vol, high corr.
pc
truncated-pc
Glasserman-Zhao
iterative pc
capc
Balland
PPR
0.0004
0.0003
0.0002
0.0001
0
-0.0001
-0.0002
-0.0003
Price of corresponding FRA
1
2
3
4
5
6
7
8
9
10
11
12
13
14
15
16
17
18
19
20
Index number of Forward



Pricing errors for at-the-money caplets
HJJ
truncated-pc
Glasserman-Zhao
iterative pc
capc
Balland
PPR
0.0006
0.0004
0.0002
0
-0.0002
-0.0004
-0.0006
Error in price
1
2
3
4
5
6
7
8
9
10
11
12
13
14
15
16
17
18
19
20
Index number of caplet


out-of-the-money caplet pricing errors
HJJ
truncated-pc
Glasserman-Zhao
iterative pc
capc
Balland
PPR
0.0004
0.0002
0
-0.0002
-0.0004
-0.0006
pricing error in corresponding caplet
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20
index number of forward


out-of-the-money caplet pricing errors
HJJ
truncated-pc
Glasserman-Zhao
iterative pc
capc
Balland
PPR
cani
0.0004
0.0002
0
-0.0002
-0.0004
-0.0006
pricing error in corresponding caplet
1
2
3
4
5
6
7
8
9
10
11
12
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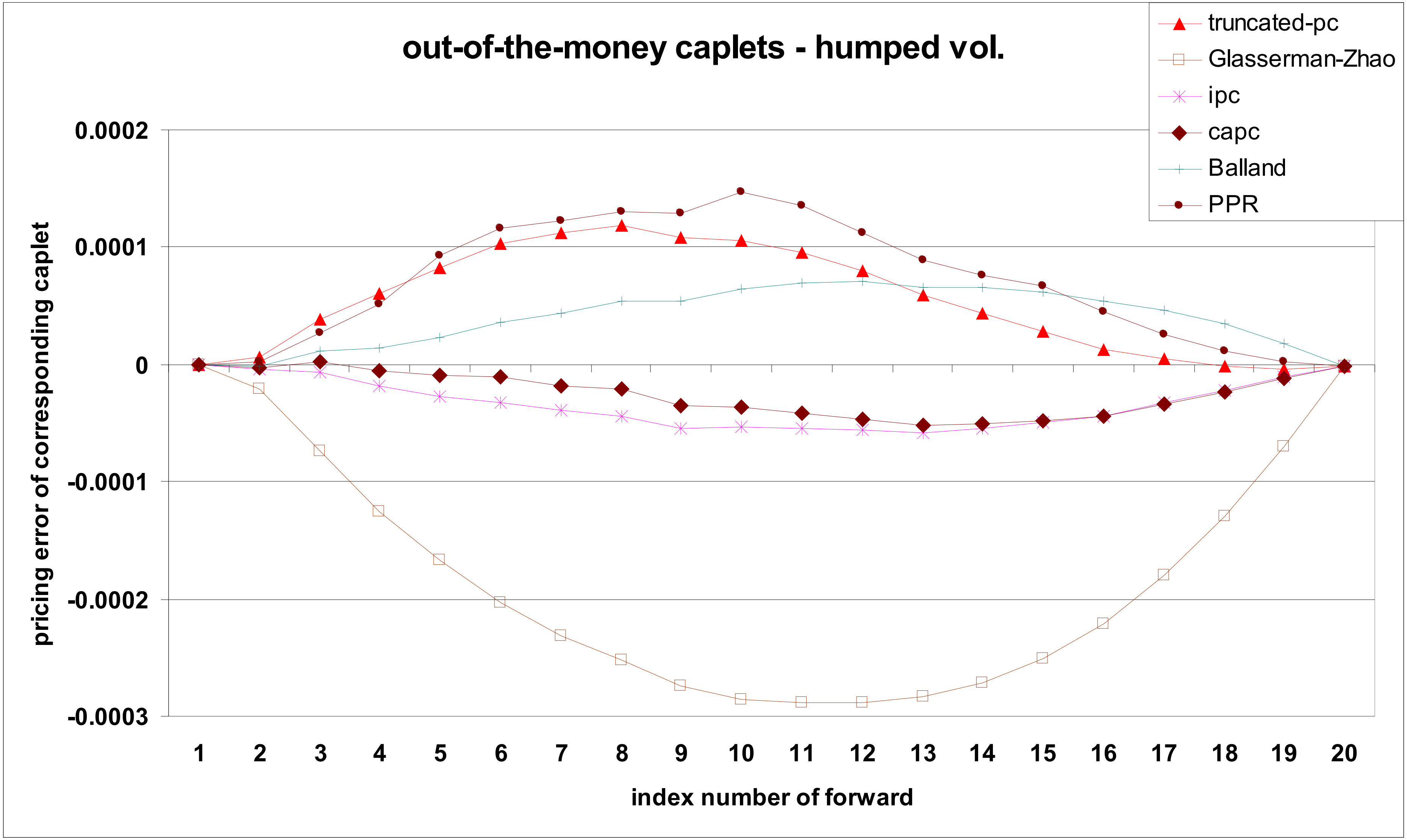
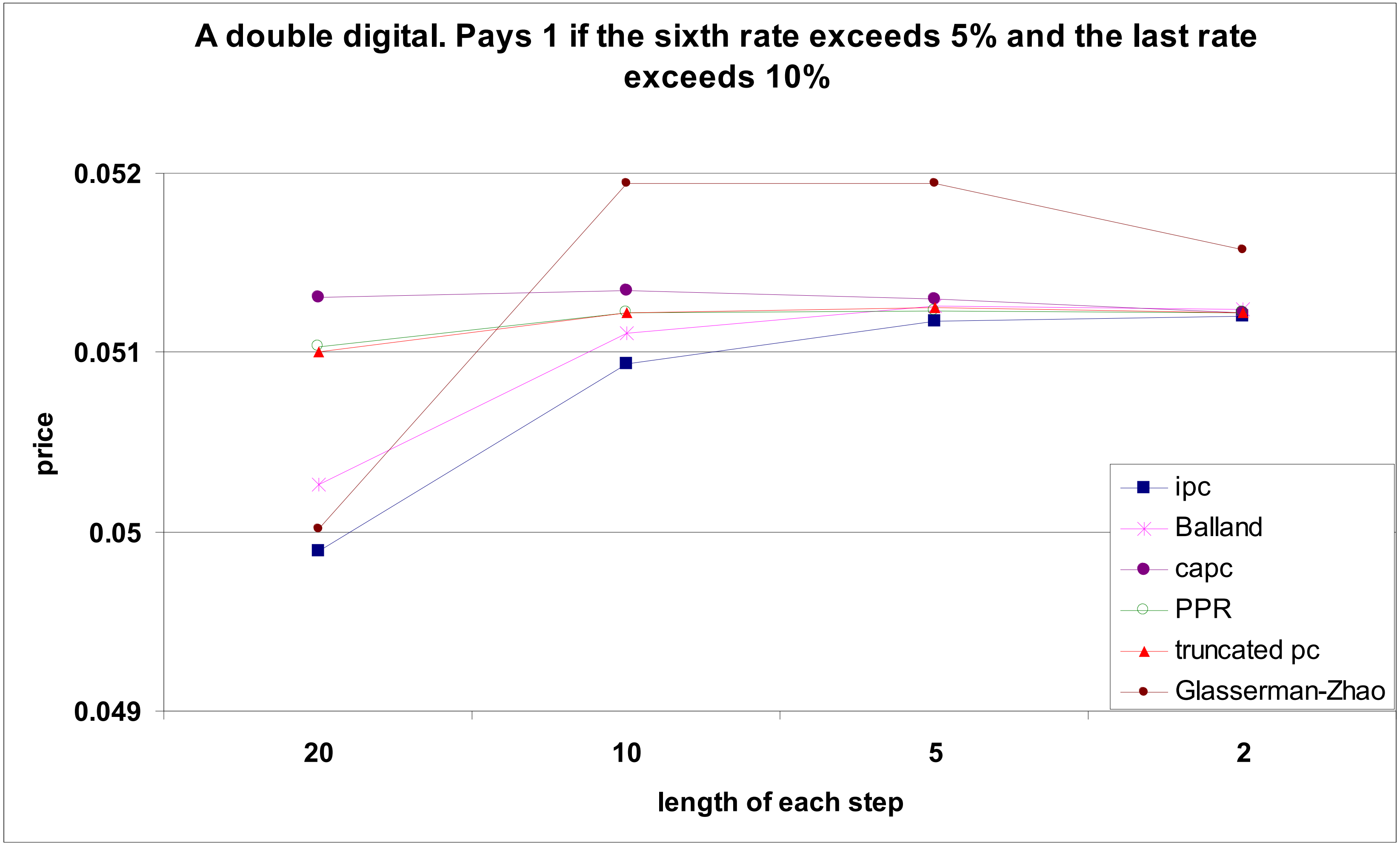
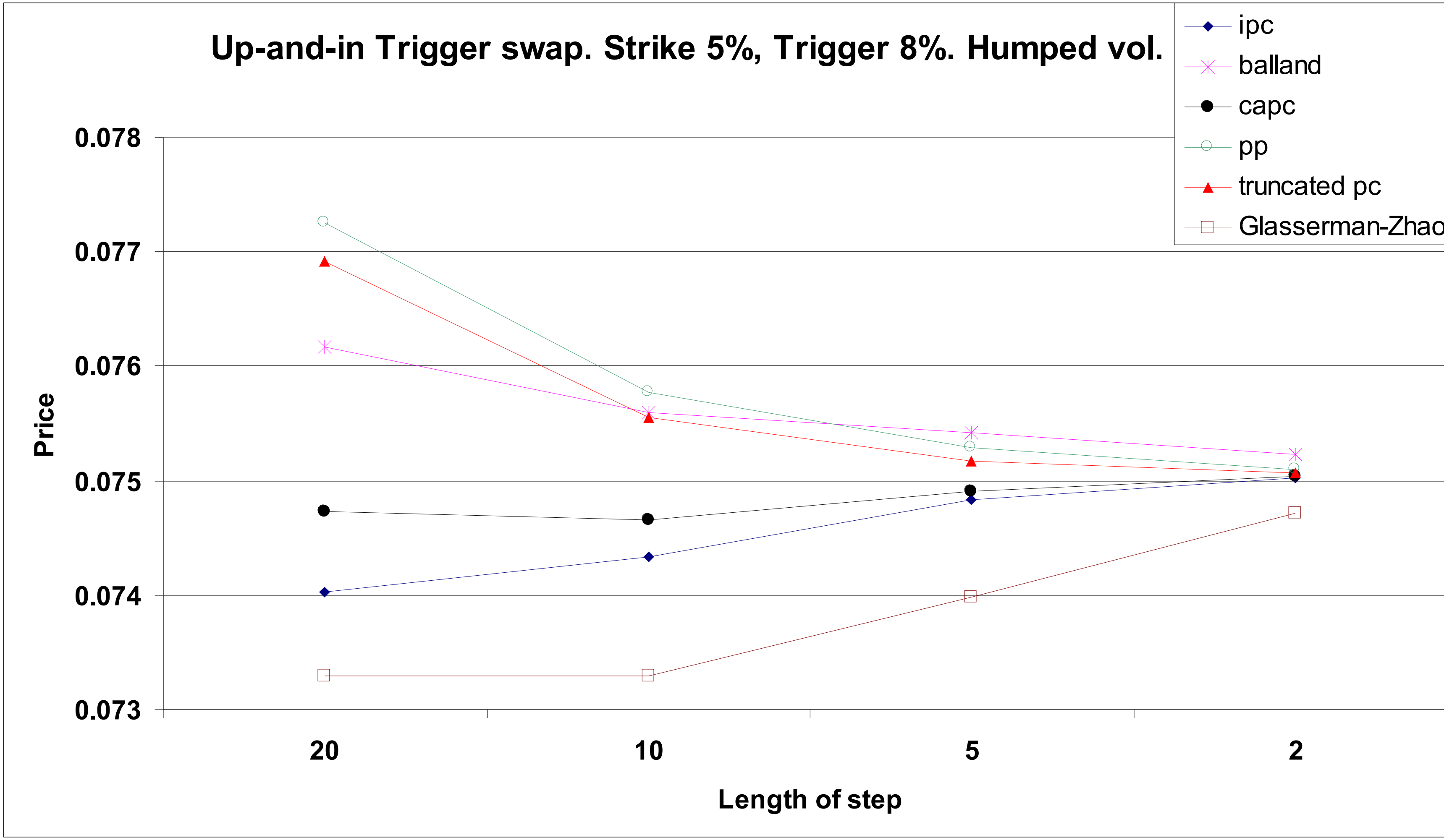
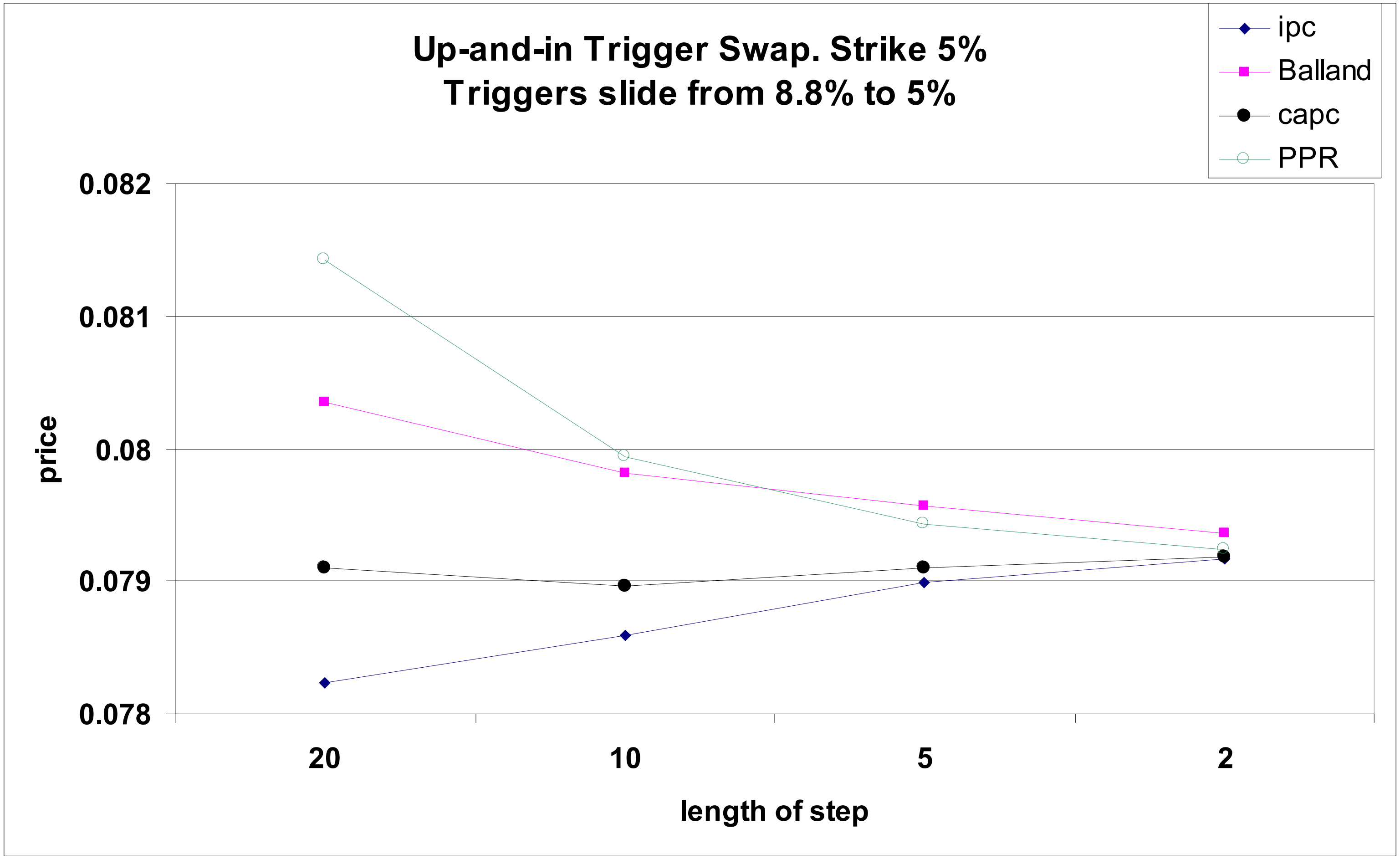
14
15
16
17
18
19
20
index number of forward


out-of-the-money caplets - humped vol.
truncated-pc
Glasserman-Zhao
ipc
capc
Balland
PPR
0.0002
0.0001
0
-0.0001
-0.0002
-0.0003
pricing error of corresponding caplet
1
2
3
4
5
6
7
8
9
10
11
12
13
14
15
16
17
18
19
20
index number of forward

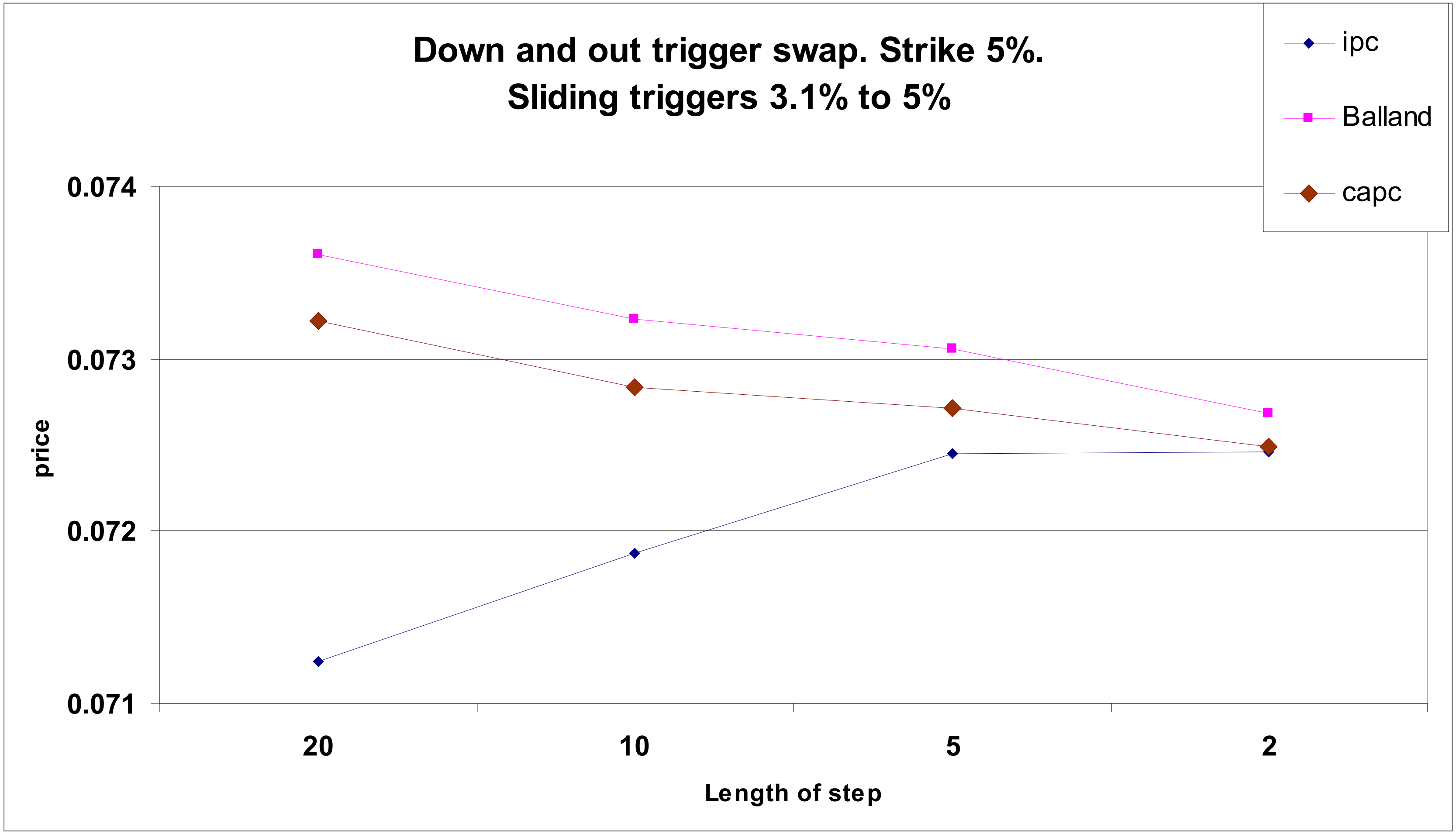

A double digital. Pays 1 if the sixth rate exceeds 5% and the last rate exceeds 10%
0.052
0.051
0.05
0.049
price
20
10
5
2
length of each step
ipc
Balland
capc
PPR
truncated pc
Glasserman-Zhao



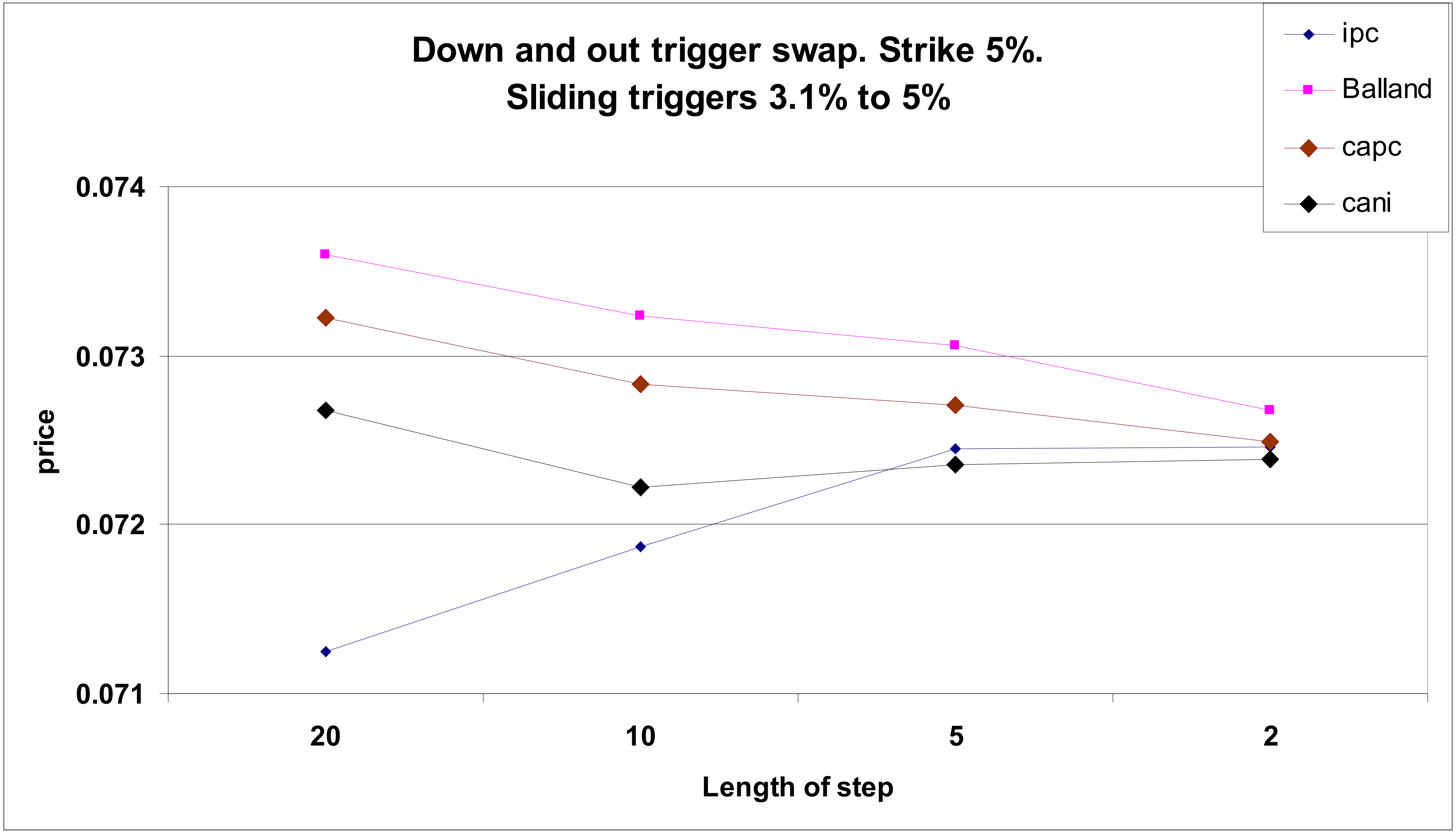
Up-and-in Trigger swap. Strike 5%, Trigger 8%. Humped vol.
ipc
balland
capc
pp
truncated pc
Glasserman-Zhao
0.078
0.077
0.076
0.075
0.074
0.073
Price
20
10
5
2
Length of step

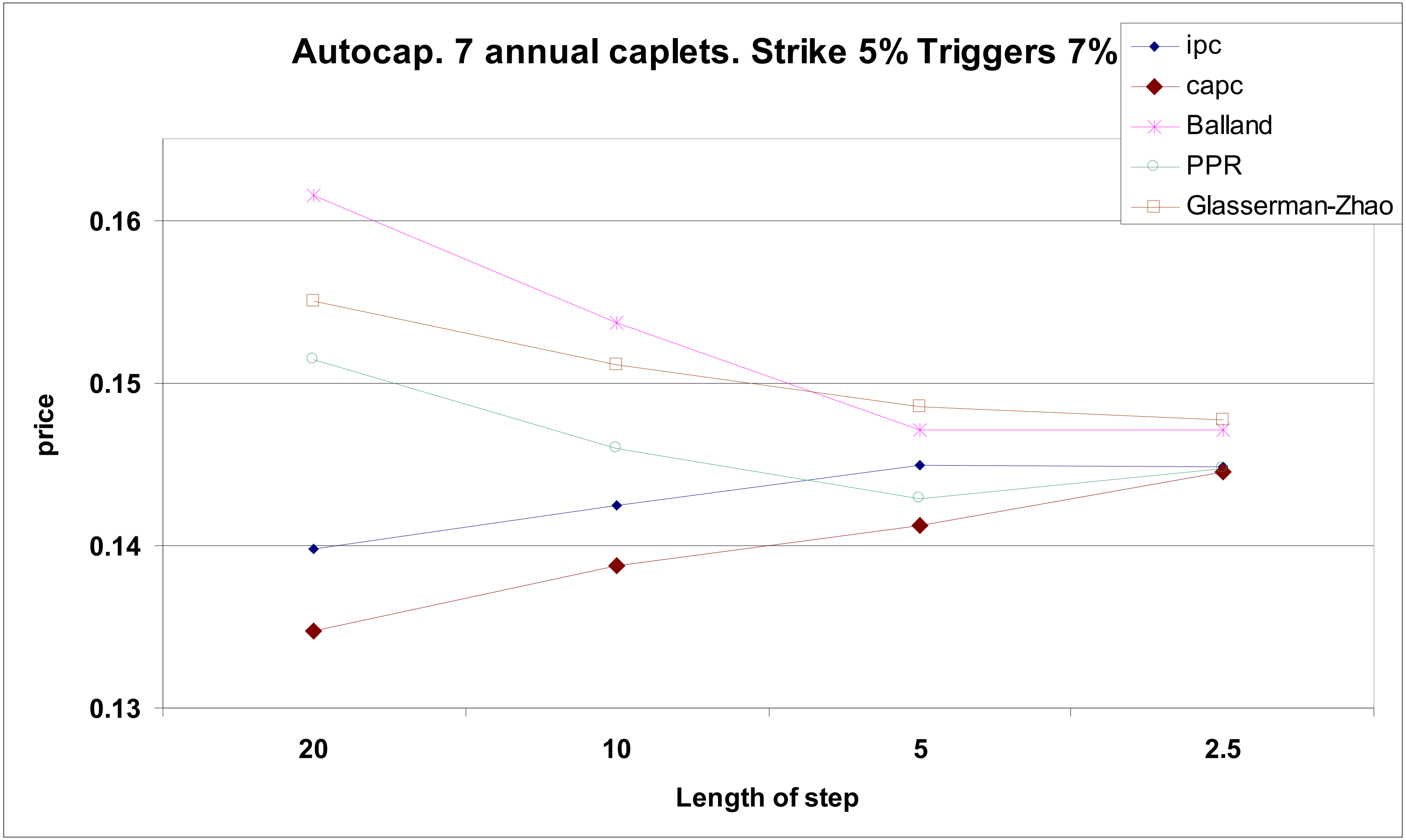


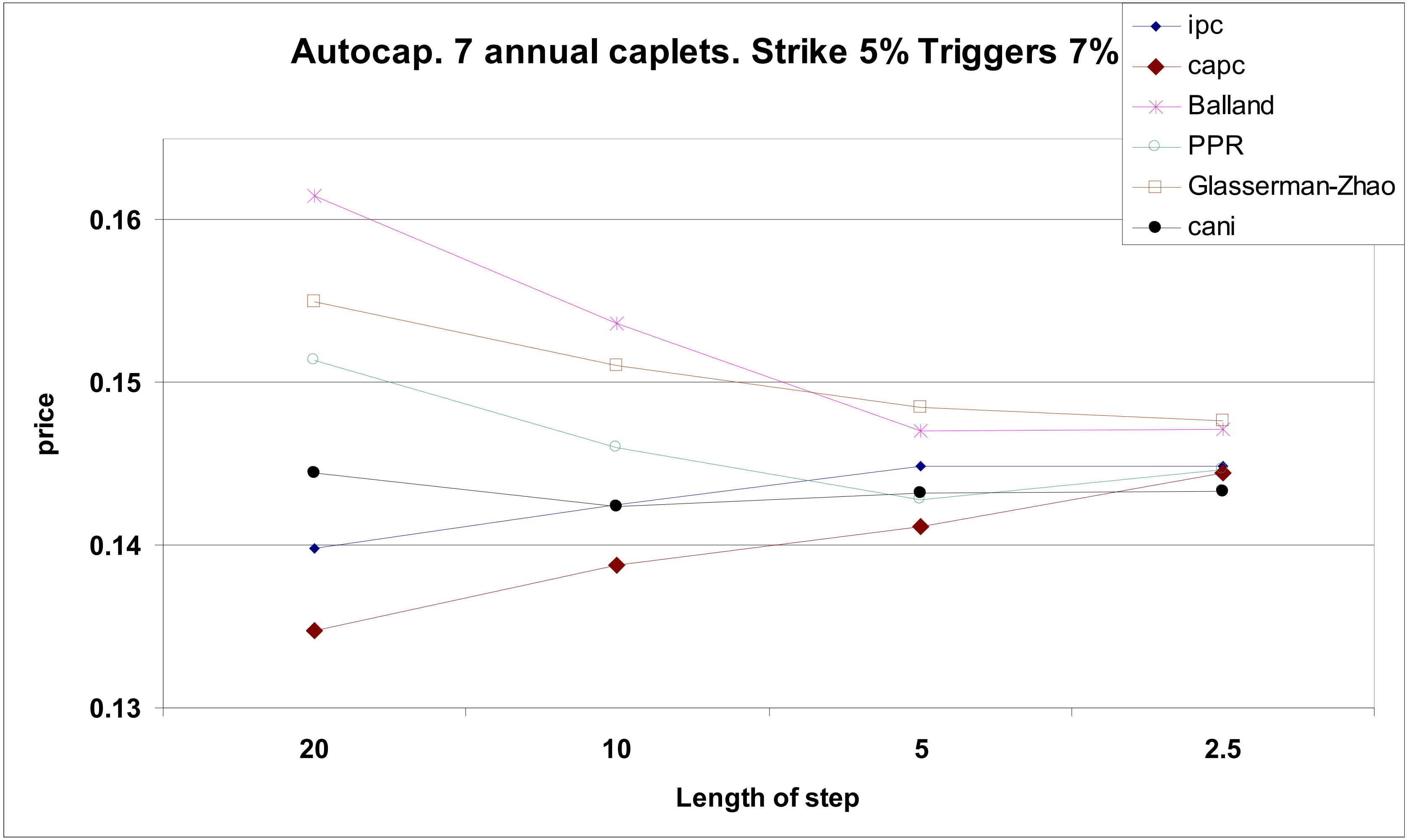
Up-and-in Trigger Swap. Strike 5%
Triggers slide from 8.8% to 5%
ipc
Balland
capc
PPR
0.082
0.081
0.08
0.079
0.078
price
20
10
5
2
length of step



Down and out trigger swap. Strike 5%.
Sliding triggers 3.1% to 5%
ipc
Balland
capc
0.074
0.073
0.072
0.071
price
20
10
5
2
Length of step


Down and out trigger swap. Strike 5%.
Sliding triggers 3.1% to 5%
ipc
Balland
capc
cani
0.074
0.073
0.072
0.071
price
20
10
5
2
Length of step


Autocap. 7 annual caplets. Strike 5% Triggers 7%
ipc
capc
Balland
PPR
Glasserman-Zhao
price
0.16
0.15
0.14
0.13
20
10
5
2.5
Length of step



Autocap. 7 annual caplets. Strike 5% Triggers 7%
ipc
capc
Balland
PPR
Glasserman-Zhao
cani
price
0.16
0.15
0.14
0.13
20
10
5
2.5
Length of step